In tali nunquam lassat venatio sylva.
A.D.1884.

FIRST PART

OF AN

ELEMENTARY TREATISE

ON

SPHERICAL TRIGONOMETRY.

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BOSTON.
JAMES MUNROE AND COMPANY.
M DCCC XXXVI.

CAMBRIDGE PRESS:
METCALF, TORRY, AND BALLOU.

gift
Tappan Presb. Ass.
12-10-1931

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CHAPTER I.

Definitions.

1. *Spherical Trigonometry* treats of the solution of *spherical triangles.*

A *Spherical Triangle* is a portion of the surface of a sphere included between three arcs of great circles.

In the present treatise those spherical triangles only are treated of, in which the sides and angles are less than 180°.

2. *The angle,* formed by two sides of a spherical triangle, is the same as the angle formed by their planes.

3. An *isosceles* spherical triangle is one which has two of its sides equal.

An *equilateral* spherical triangle is one which has all its sides equal.

4. A spherical *right* triangle is one which has a right angle, all other spherical triangles are called *oblique.*

We shall in spherical trigonometry, as we did in

plane trigonometry, attend first to the solution of right triangles.

CHAPTER II.

Spherical Right Triangles.

SECTION. I.

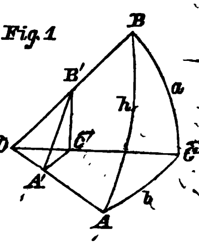
Napier's Rules for the Solution of Spherical Right Triangles.

5. *Problem.* To investigate some relations between the sides and angles of a spherical right triangle.

Solution. The importance of this problem is obvious; for, unless some relations were known between the sides and the angles, they could not be determined from each other, and there could be no such thing as the solution of a spherical triangle.

Let, then, ABC (fig. 1.) be a spherical right triangle, right angled at C. Call the hypothenuse AB, h; and call the legs BC and
(427) AC, opposite the angles A and B, respectively a and b.

Fig. 1



Let O be the centre of the sphere. Join OA, OB, OC.

The angle A is, by art. 2, equal to the angle of the planes BOA and COA. The angle B is equal to the angle of the planes BOC and BOA. The angle

of the planes BOC and AOC is equal to the angle (428)
C, that is, to a right angle; these two planes are, therefore, perpendicular to each other.

Moreover, the angle BOA, measured by BA, is equal to BA or h; BOC is equal to its measure BC (429)
or a, and AOC is equal to its measure AC or b.

Through any point A' of the line OA, suppose a plane to pass perpendicular to OA. Its intersections $A'C'$ and $A'B'$, with the planes COA and BOA must (430)
be perpendicular to OA', because they are drawn through the foot of this perpendicular.

As the plane $B'A'C'$ is perpendicular to OA, it must be perpendicular to AOC; and its intersection $B'C'$, with the plane BOC, which is also perpendicular to AOC, must likewise be perpendicular to AOC. Hence $B'C'$ must be perpendicular to $A'C'$ and OC' which pass through its foot in the plane AOC.

All the triangles $A'OB'$, $A'OC'$, $B'OC'$, and (431)
$A'B'C'$ are then right-angled; and the comparison of them leads to the desired equations, as follows:

First. We have from triangle $A'OB'$ by (5) and (429),

$$\cos.\ A'OB' = \cos.\ h = \frac{OA'}{OB'}; \tag{432}$$

and from triangles $A'OC'$ and $B'OC'$

$$\cos.\ A'OC' = \cos.\ b = \frac{OA'}{OC'}, \tag{433}$$

$$\cos.\ B'OC' = \cos.\ a = \frac{OC'}{OB'}. \tag{434}$$

The product of the two last equations is

$$\cos.\ a \cos.\ b = \frac{OA'}{OC'} \times \frac{OC'}{OB'} = \frac{OA'}{OB'}; \tag{435}$$

hence from the equality of the second members of equations, (432) and (435),

(436) $$\cos. h = \cos. a \cos. b.$$

Secondly. From triangle $A'B'C'$ we have by (5)
(437) and (428), and the fact that the angle $B'A'C'$ is equal to the inclination of the two planes BOC and BOA,

(438) $$\cos. B'A'C' = \cos. A = \frac{A'C'}{A'B'};$$

and, from triangles $A'OC'$ and $A'OB'$, by (5) and (429),

(439) $$\text{tang. } C'OA' = \text{tang. } b = \frac{A'C'}{A'O},$$

(440) $$\text{cotan. } B'OA' = \text{cotan. } h = \frac{A'O}{A'B'}.$$

The product of (439) and (440) is

(441) $$\text{tang. } b. \text{ cotan. } h = \frac{A'C'}{A'O} \times \frac{A'O}{A'B'} = \frac{A'C'}{A'B'};$$

hence, by (438),

(442) $$\cos. A = \text{tang. } b. \text{ cotan. } h.$$

Thirdly. Corresponding to the preceding equation between the hypothenuse h, the angle A, and the adjacent side b, there must be a precisely similar equation between the hypothenuse h, the angle B, and the adjacent side a; which is

(443) $$\cos. B = \text{tang. } a \text{ cotan. } h.$$

Fourthly. From triangles $B'OC'$, $B'OA'$, and $B'A'C'$, by (5), (429), and (437),

(444) $$\sin. B'OC' = \sin. a = \frac{B'C'}{OB'}.$$

$$\sin. B'OA' = \sin. h = \frac{B'A'}{OB'}, \tag{445}$$

$$\sin. B'A'C' = \sin. A = \frac{B'C'}{B'A'}. \tag{446}$$

The product of (445) and (446) is

$$\sin. h \sin. A = \frac{B'A'}{OB'} \times \frac{B'C'}{B'A'} = \frac{B'C'}{OB'}; \tag{447}$$

hence, by (444),

$$\sin. a = \sin. h \sin. A. \tag{448}$$

Fifthly. The preceding equation between h, the angle A, and the opposite side a, leads to the following corresponding one between h, the angle B, and the opposite side b;

$$\sin. b = \sin. h \sin. B. \tag{449}$$

Sixthly. From triangles $C'OA'$, $B'A'C'$ and $B'OC'$, by (5), (429), and (437),

$$\sin. C'OA' = \sin. b = \frac{A'C'}{OC'}, \tag{450}$$

$$\text{cotan. } B'A'C' = \text{cotan. } A = \frac{A'C'}{B'C'}, \tag{451}$$

$$\text{tang. } B'OC' = \text{tang. } a = \frac{B'C'}{OC'}. \tag{452}$$

The product of (451) and (452) is

$$\text{cotan. } A \text{ tang. } a = \frac{A'C'}{B'C'} \times \frac{B'C'}{OC'} = \frac{A'C'}{OC'}; \tag{453}$$

hence, by (450),

$$\sin. b = \text{cotan. } A \text{ tang. } a. \tag{454}$$

Seventhly. The preceeding equation between the angle A, the opposite side a, and the adjacent side b, leads to the following corresponding one between

the angle B, the opposite side b, and the adjacent side a;

(455) $$\sin. a = \text{cotan.}\, B \,\text{tang.}\, b.$$

Eighthly. From (10),

(456) $$\text{tang.}\, a = \frac{\sin. a}{\cos. a},$$

(457) $$\text{tang.}\, b = \frac{\sin. b}{\cos. b};$$

which, substituted in (454) and (455), gives

(458) $$\sin. a = \frac{\text{cotan.}\, B \sin. b}{\cos. b},$$

(459) $$\sin. b = \frac{\text{cotan.}\, A \sin. a}{\cos. a}.$$

Multiplying (458) by $\cos. b$ and (459) by $\cos. a$, we have

(460) $$\sin. a \cos. b = \text{cotan.}\, B \sin. b,$$

(461) $$\sin. b \cos. a = \text{cotan.}\, A \sin. a.$$

The product of (460) and (461) is

(462) $$\sin. a \sin. b \cos. a \cos. b = \cot. A \cot. B \sin. a \sin. b;$$

which, divided by $\sin. a \sin. b$, becomes

(463) $$\cos. a \cos. b = \text{cotan.}\, A \,\text{cotan.}\, B.$$

But, by (436),

(464) $$\cos. h = \cos. a \cos. b;$$

hence

(465) $$\cos. h = \text{cotan.}\, A \,\text{cotan.}\, B.$$

Ninthly. We have, by (436) and (449),

(466) $$\cos. a = \frac{\cos. h}{\cos. b},$$

(467) $$\sin. B = \frac{\sin. b}{\sin. h},$$

the product of which is by (10) and (11)

$$\cos. a \sin. B = \frac{\sin. b \cos. h}{\cos. b \sin. h} = \frac{\sin. b}{\cos. h} \cdot \frac{\cos. h}{\sin. h}$$

$$= \text{tang.}\ b \text{ cotan. } h. \tag{468}$$

But, by (442),

$$\cos. A = \text{tang.}\ b \text{ cotan. } h; \tag{469}$$

Hence, from the equality of the second members of (468) and (469),

$$\cos. A = \cos. a \sin. B. \tag{470}$$

Tenthly. The preceding equation between the side a, the opposite angle A, and the adjacent angle B, leads to the following similar one between the side b, the opposite angle B, and the adjacent angle A;

$$\cos. B = \cos. b \sin. A. \tag{471}$$

6. *Corollary.* The ten equations, (436), (442), (443), (448), (449), (454), (455), (465), (470), and (471), have, by a most happy artifice, been reduced to two very simple theorems, called, from their celebrated inventor, *Napier's Rules.* (472)

In these rules, the complements of the hypothenuse and the angles are used instead of the hypothenuse and the angles themselves, and the right angle is neglected.

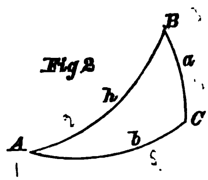
Of the five parts, then, the legs, the complement of the hypothenuse and the complements of the angles; either part may be called the *middle part.* The two parts, including the middle part on each side, are called the *adjacent parts;* and the other two parts are called the *opposite parts.* The two theorems are as follows: (473)

(474) I. ***The sine of the middle part is equal to the product of the tangents of the two adjacent parts.***

(475) II. ***The sine of the middle part is equal to the product of the cosines of the two opposite parts.***

Demonstration. To demonstrate the preceding rules, it is only necessary to compare all the equations which can be deduced from them, with those previously obtained (472).

Let there be the spherical right triangle ABC (fig. 2.) right-angled at C.



First. If co. h were made the middle part, then, by (473), co. A and co. B would be adjacent parts, and a and b opposite parts; and, by (474) and (475), we should have

(476) $$\sin. (\text{co. } h) = \text{tang. } (\text{co. } A) \text{ tang. } (\text{co. } B),$$

(477) $$\sin. (\text{co. } h) = \cos. a \cos. b;$$

or

(478) $$\cos. h = \text{cotan. } A \text{ cotan. } B,$$

(479) $$\cos. h = \cos. a \cos. b;$$

which are the same as (465) and (436).

Secondly. If co. A were made the middle part; then, by (473), co. h and b would be adjacent parts, and co. B and a opposite parts; and, by (474) and (475), we should have

(480) $$\sin. (\text{co. } A) = \text{tang. } (\text{co. } h) \text{ tang. } b,$$

(481) $$\sin. (\text{co. } A) = \cos. (\text{co. } B) \cos, a;$$

or

(482) $$\cos. A = \text{cotan. } h \text{ tang. } b,$$

$$\cos. A = \sin. B \cos. a; \tag{483}$$

which are the same as (442) and (470).

In like manner, if co. B were made the middle part, we should have

$$\cos. B = \text{cotan. } h \text{ tang. } a, \tag{484}$$

$$\cos. B = \sin. A \cos. b; \tag{485}$$

which are the same as (443) and (471).

Thirdly. If a were made the middle part; then, by (473), co. B and b would be the adjacent parts, and co. A and co. h the opposite parts; and, by (474) and (475), we should have

$$\sin. a = \text{tang. (co. } B) \text{ tang. } b, \tag{486}$$

$$\sin. a = \cos. (\text{co. } A) \cos. (\text{co. } h); \tag{487}$$

or

$$\sin. a = \text{cotan. } B \text{ tang. } b, \tag{488}$$

$$\sin. a = \sin. A \sin. h; \tag{489}$$

which are the same as (455) and (448).

In like manner, if b were made the middle part, we should have

$$\sin. b = \text{cotan. } A \text{ tang. } a, \tag{490}$$

$$\sin. b = \sin. B \sin. h; \tag{491}$$

which are the same as (454) and (449).

Having, thus, made each part successively the middle part, the ten equations, which we have obtained, must be all the equations included in (474) and (475); and we perceive that they are identical with the ten equations of (472).

7. *Corollary.* When h is less than $90°$, the first member of (436),

$$\cos. h = \cos. a \cos b, \tag{492}$$

is positive; and therefore the factors of its second
(493) member must either be both positive or both negative; that is, the two legs a and b must, by the following Lemma (496), be both greater or both less than 90°.

But when h is greater than 90°, the first member of (492) is by (496) negative; and therefore one of
(494) the factors of the second member must be positive, while the other is negative; that is, of the two legs a and b, one must be less while the other is greater than 90°.

These results may be simply expressed as follows:

The three sides of a spherical right triangle are
(495) *either all less than* 90°; *or else, one is less while the other two are greater than* 90°; unless one of them is equal to 90° as in (522).

8. *Lemma.* The sine and cosecant of an angle,
(496) which is greater than 90° and less than 180°, are positive; but its cosine, tangent, cotangent, and secant are negative.

Demonstration. Let the given angle be $90° + N$, N being less than 90°. Then $90° + N$ and $90° -$
(497) N are supplements of each other, since their sum is equal to 180°, and we have from (195) and (5),

(498) $\sin.\ (90° + N) = \sin.\ (90° - N) = \cos.\ N$

(499) $\cos.\ (90° + N) = - \cos.\ (90° - N) = - \sin.\ N$

(500) $\text{tang.}\ (90° + N) = - \text{tang.}\ (90° - N) = - \text{cotan.}\ N$

(501) $\text{cotan.}\ (90° + N) = - \text{cotan.}\ (90° - N) = - \tan.\ N$

(502) $\sec.\ (90° + N) = - \sec.\ (90° - N) = - \text{cosec.}\ N$

(503) $\text{cosec.}\ (90° + N) = \text{cosec.}\ (90° - N) = \sec.\ N$

all which equations agree with (496).

9. *Corollary.* The equation (436)

$$\cos. h = \cos. a \cos. b, \tag{504}$$

leads also to the result, that *the hypothenuse differs* (505) *less from* 90° *than does either of the legs*, the case of either side equal to 90° being excepted.

Demonstration. The factors cos. a and cos. b of the second member of the above equation are, by (5), fractions whose numerators are less than their denominators. Their product, neglecting the signs, must then be less than either of them, as cos. a for instance, or

$$\cos h < \cos a\,; \tag{506}$$

and therefore h must differ less from 90° than a does, as is evident from the following Lemma.

10. *Lemma.* Of angles less than 180°, the one which differs the least from 90° has the largest sine, tangent, and secant; and the smallest cosine, cotan- (507) gent, and cosecant; no regard being had to the signs.

Demonstration. Let the quantity by which an angle differs from 90° be N; and the angle is either 90° (508) $+ N$ or $90° - N$. But, by (498),

$$\sin. (90° + N) = \sin. (90° - N). \tag{509}$$

Now the smaller N is, the larger must $(90° - N)$ be, and by (30′) the larger the sine of $(90° - N)$; (510) that is, the less the angle differs from 90° the larger is its sine.

Again, by (499),

$$\cos. (90° + N) = -\cos. (90° - N) = -\sin. N. \tag{511}$$

Now the smaller N is, the smaller, by (30′), must its

(512) sine be, since it is less than 90°; and therefore the smaller, neglecting the signs, must the cosine of the given angle be.

In the same way, by means of (48′), the proposition (507) might be proved, with regard to the tangents, cotangents, secants, and cosecants. Indeed, it readily appears from the equations (13), (7), and (10)
(513) that the sine, tangent, and secant of an angle increase, while the cosine, cosecant, and cotangent diminish.

11. *Corollary.* When A is less than 90°, the first member of (470)

(514) $$\cos. A = \cos. a \ \sin. B$$

is positive, and therefore the factor $\cos. a$ of the sec-
(515) ond member, being multiplied by the positive factor sin. B (496), must be positive; that is, a must be less than 90°. But, if A is greater than 90°, the first member of (514) is, by (496), negative, and therefore
(516) the factor $\cos. a$ of the second member must be negative; that is, a must, by (496), be greater than 90°. We may express this result as follows:

An angle and its opposite leg in a spherical right
(517) *triangle must be both less or both greater than* 90°, *or by* (522) *both equal to* 90°.

12. *Corollary.* The equation (470)

(518) $$\cos. A = \cos. a \ \sin. B,$$

leads also to the result that *an angle differs less from* 90° *than its opposite leg*, the case of either side, equal to 90°, being excepted.

Demonstration. Since the second member of (518) is the product of the two fractions cos. a and sin. B, (520) the first member must be less than either of them. Thus, neglecting the sines,

$$\cos. A < \cos. a; \qquad (521)$$

hence, by (507), A differs less from 90° than does a.

13. *Corollary.* When, in a spherical right triangle, either side is equal to 90°, one of the other two sides (522) is also equal to 90°; and each side is equal to its opposite angle.

Demonstration. First. If either of the legs is equal to 90°, the corresponding factor of the second (523) member of (436),

$$\cos. h = \cos. a \cos b, \qquad (524)$$

is, by (157), equal to zero; which gives

$$\cos. h = 0, \qquad (525)$$

or, by (157),

$$h = 90°. \qquad (526)$$

Again, if we have

$$h = 90°, \qquad (527)$$

it follows, from (157) and (524), that

$$0 = \cos. a \cos. b, \qquad (528)$$

and therefore either cos. a or cos. b must be zero; that is, either a or b must be equal to 90°. (529)

Secondly. When either side is equal to 90°, it follows, from (523) and (526), that

$$h = 90°. \qquad (530)$$

This result, substituted in equation (448),

$$\sin. a = \sin. h \sin. A, \qquad (531)$$

produces, by (158),

(532) $$\sin. a = \sin. A;$$

which gives

(533) $$a = A;$$

because, from (517), a could not be equal to the supplement of A.

14. *Corollary.* When both the legs of a spherical
(534) right triangle are equal to 90°, all the sides and angles are, from (523), (526), and (533), also equal to 90°.

15. *Corollary.* When two of the angles of a
(535) spherical triangle are equal to 90°, the opposite sides are also equal to 90°.

Demonstration. For, in this case, one of the factors of the second member of the equation (465),

(536) $$\cos. h. = \text{cotan. } A. \text{ cotan. } B,$$

must, by (159), be equal to zero, since either A or B is 90°; hence

(537) $$\cos. h. = 0,$$

or, by (157),

(538) $$h. = 90°,$$

and the remainder of the proposition follows from (522).

16. *Corollary.* When all the angles of a spherical
(539) right triangle are equal to 90°, all the sides are also, by (535), equal to 90°.

17. *Corollary.* *The sum of the angles of a spher-*
(540) *ical triangle is greater than* 180°, *and less than* 360°; *and each angle is less than the sum of the other two.*

Demonstration. First Case. When each of the legs differs from 90°, the equation (470),

$$\cos. A = \cos. a \ \sin. B, \tag{541}$$

gives, by (520),

$$\cos. A < \sin. B; \tag{542}$$

or, by (5),

$$\sin. (90° - A) < \sin. B. \tag{543}$$

First. The only case in which it is necessary to prove that the sum of the angles is greater than 180°, or, that the sum of A and B is greater than 90°, is, when A and B are both acute. In this case, by (30′) and (543),

$$B > 90° - A; \tag{544}$$

or

$$A + B > 90°. \tag{545}$$

Secondly. As the preceding equation expresses, that when the right angle is the greatest angle of the triangle it is less than the sum of the other two angles; (546)
we have only to show farther, that, when either of the other angles, as B, is the greatest angle, and of course obtuse, it is less than the sum of the other two angles. We may suppose A to be acute. Then, as the difference between B and 90° is $B - 90°$, and as that be- (547)
tween 90° and $90° - A$ is $90° - (90° - A)$ or A; we have, by (507) and (543),

$$B - 90° < A, \tag{548}$$

or

$$B < 90° + A; \tag{549}$$

from which we conclude that each angle of a right triangle is less than the sum of the other two. (550)

Thirdly. The only case in which it is necessary to prove that the sum of all the angles is less than
(551) 360°, or that the sum of A and B is less than 270°, is when A and B are both obtuse. But, if A is obtuse, $90° - A$ is the negative of $A - 90°$, which may by (202) be substituted for it in (543), and we have

(552) $$\sin. (A - 90°) < \sin. B;$$

whence, by (507),

(553) $$B - 90° < 90° - (A - 90°),$$

or

(554) $$B - 90° < 180° - A,$$

or

(555) $$A + B < 270°.$$

Second Case. When one of the legs is equal to 90°, its opposite angle is also 90°, by (522); and therefore whatever is the value of the third angle, it cannot but satisfy the conditions of the proposition (540).

SECTION. II.

Solution of Spherical Right Triangles.

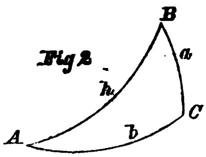
18. To solve a spherical right triangle, two parts must be known in addition to the right angle. From the two known parts, the other three parts are to be determined, separately, by equations derived from Napier's Rules. The two given parts with the one to be determined are, in each case, to enter into the same equation. *These three parts are either all adjacent to each other, in which case the middle one is*
(556) *taken as the* MIDDLE PART, *and the other two are, by*

(473), ADJACENT PARTS ; *or one is separated from the other two, and then the part, which stands by itself, is the* MIDDLE PART, *and the other two are, by* (473), OPPOSITE PARTS.

19. *Problem.* To solve a spherical right triangle, when the hypothenuse and one of the angles are known.

Solution. Let ABC (fig. 2.) be the right triangle, right angled at C; and let the sides be denoted as in (427). Let h and A be given, to solve the triangle.

Fig 2.



First. To find the other angle B. The three parts which are to enter into the same equation are co. h, co. A, and co. B; and, by (556), as they are all adjacent to each other, co. h is the middle part, and co. A and co. B are adjacent parts. Hence, by (474),

$$\left\{\begin{array}{l} \sin.\ (\text{co. } h) = \text{tang. } (\text{co. } A)\ \text{tang. } (\text{co. } B), \\ \text{or} \\ \cos.\ h = \text{cotan. } A\ \text{cotan. } B; \end{array}\right. \quad (557)$$

and, by (7),

$$\text{cotan. } B = \frac{\cos.\ h}{\text{cotan. } A} = \cos.\ h\ \text{tang. } A. \quad (558)$$

Secondly. To find the opposite leg a. The three parts are co. A, co. h, and a; of which, by (556), a is the middle part, and co. h and co. A are the opposite parts. Hence, by (475),

$$\left\{\begin{array}{l} \sin.\ a = \cos.\ (\text{co. } h)\ \cos.\ (\text{co. } A), \\ \text{or} \\ \sin.\ a = \sin.\ h\ \sin.\ A. \end{array}\right. \quad (559)$$

Thirdly. To find the adjacent leg b. The three parts are co. A, co. h, and b; of which co. A is the middle part, and co. h and b are the adjacent parts. Hence, by (474),

(560) $$\left\{\begin{array}{l} \sin. (\text{co. } A) = \text{tang. } (\text{co. } h) \text{ tang } b, \\ \text{or} \\ \cos. A = \text{cotan. } h \text{ tang. } b; \end{array}\right.$$

and, by (7),

(561) $$\text{tang. } b = \frac{\cos. A}{\text{cotan. } h} = \text{tang. } h \cos. A.$$

20. *Scholium.* The tables always give two angles, which are supplements of each other, corre-
(562) sponding to each sine, cosine, &c. But it is easy to choose the proper angle for the particular case, by referring to (495) and (517); or by having regard to the signs of the different terms of the equation, as determined by (496).

21. *Scholium.* When h and A are both equal to 90°, the values of cotan. B and tang. b (558) and (561), are indeterminate; since the numerators and denominators of the fractional values are, by (157)
(563) and (159), equal to zero; and in this case there are an infinite number of triangles which satisfy the given values of h and A.

The problem is impossible by (535) or (538), if the
(564) given value of h differs from 90° while that of A is equal to 90°.

EXAMPLES.

1. Given in the spherical right triangle (fig. 2.), $h = 145°$ and $A = 23° \, 28'$; to solve the triangle.

Solution.

	By (558),	by (559),	by (561),
h,	cos. 9.91336 n,*	sin. 9.75859,	tang. 9.84523 n.
A,	tang. 9.63761,	sin. 9.60012,	cos. 9.96251
B,	cotan. 9.55097 n;	a sin. 9.35871;	b tang. 9.80774 n.

Ans. $B = 109° \; 34'$, $a = 13° \; 12'$, $b = 147° \; 17'$.

2. Given, in the spherical right trangle (fig. 2.), $h = 32° \; 34'$ and $A = 44° \; 44'$, to solve the triangle.

Ans. $B = 50° \; 8'$,
$a = 22° \; 16'$,
$b = 24° \; 24'$.

22. ***Problem.*** To solve a spherical right triangle, when its hypothenuse and one of its legs are known.

Solution. Let ABC (fig. 2.) be the triangle; h the given hypothenuse, and a the given leg.

First. To find the opposite angle A; a is the middle part, and co. A and co. h are the opposite parts. Hence, by (475),

$$\left\{\begin{array}{l} \sin. a = \cos. (\text{co. } h) \cos. (\text{co. } A); \\ \text{or} \\ \sin. a = \sin. h \sin A; \end{array}\right. \tag{565}$$

and by (7),

$$\sin. A = \frac{\sin. a}{\sin. h} = \sin. a \text{ cosec. } h. \tag{566}$$

* The letter n placed after a logarithm indicates it to be the logarithm of a negative quantity, and it is plain that when the number of such logarithms to be added together is even, the sum is the logarithm of a positive quantity; but if odd, the sum is the logarithm of a negative quantity.

Secondly. To find the adjacent angle B; co. B is the middle part, and co. h and a are the adjacent parts. Hence, by (474),

(567) $$\left\{\begin{array}{l} \text{sin. (co. } B.) = \text{tang. } a \text{ tang. (co. } h), \\ \text{or} \\ \text{cos. } B = \text{tang. } a \text{ cotan. } h. \end{array}\right.$$

Thirdly. To find the other leg b; co. h is the middle part, and a and b are the opposite parts. Hence, by (475),

(568) $$\text{cos. } h = \text{cos. } a \text{ cos. } b;$$

and, by (7),

(569) $$\text{cos. } b = \frac{\text{cos. } h}{\text{cos. } a} = \text{sec. } a \text{ cos. } h.$$

23. *Scholium.* The question is impossible by (505),
(570) when the given value of the hypothenuse differs more from 90° than that of the leg.

(570' 24. *Solution.* When h and a are both equal to 90°, it may be shown, as in (563), that the values of B and b are indeterminate.

Example. Given, in the spherical right triangle (fig. 2), $a = 141° \ 11'$, and $h = 127° \ 12'$; to solve the triangle.

Ans. $A = 128° \ 7'$,
$B = 52° \ 22'$,
$b = 39° \ 6'$.

25. *Problem.* To solve a spherical right triangle, when one of its legs and the opposite angle are known.

Solution. Let ABC (fig. 2.) be the triangle; a the given leg, and A the given angle.

First. To find the hypothenuse h; a is the middle part, and co. h and co. A are the opposite parts. Hence, by (475),

$$\sin.\ a = \sin.\ h \sin.\ A; \tag{571}$$

and, by (7),

$$\sin.\ h = \frac{\sin.\ a}{\sin.\ A} = \sin.\ a \operatorname{cosec.} A. \tag{572}$$

Secondly. To find the other angle B; co. A is the middle part, and a and co. B are the opposite parts. Hence, by (475),

$$\cos.\ A = \cos.\ a \sin.\ B; \tag{573}$$

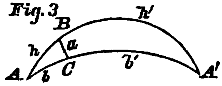
and, by (7),

$$\sin.\ B = \frac{\cos.\ A}{\cos.\ a} = \sec.\ a \cos.\ A, \tag{574}$$

Thirdly. To find the other leg b; b is the middle part, and a and co. A are the adjacent parts. Hence, by (474),

$$\sin.\ b = \operatorname{tang.} a \operatorname{cotan.} A. \tag{575}$$

Fig. 3



26. *Scholium.* There are two triangles ABC and $A'BC$ (fig. 3.), formed by producing the sides AB and AC, to the point of meeting A', both of which satisfy the conditions of the problem. For the side BC or a, and the angle A, or by art. 2 its equal A', belong to both the triangles. (576)

Now ABA' and ACA' are semicircumferences, since the line AA' joining their points of intersection

(577) is the line of intersection of their planes, and therefore passes through the centre of the sphere and is a diameter. Hence h', the hypothenuse of $A'BC$, is the
(578) supplement of h; b' is the supplement of b; and $A'BC$ is the supplement of ABC. One set of values, then, of the unknown quantities, given by the tables, as in (562), correspond to the triangle ABC, and the other set to $A'BC$.

(579) 27. *Corollary.* When the given values of a and A are equal, (572), (574), and (575) become

(580) $$\sin. h = 1 \qquad \sin. B = 1, \qquad \sin. b = 1;$$

or, by (158),

(581) $$h = 90^\circ, \qquad B = 90^\circ, \qquad b = 90^\circ;$$

as in (522).

(582) 28. *Corollary.* When a and A are equal to 90°, the values of b and B are indeterminate, as in (563).

29. *Scholium.* The problem is, by (517), impossible, when the given values of the leg and its opposite angle are such that one surpasses 90° while the other
(583) does not, or that one is equal to 90° while the other differs from 90°; and, by (519), it is impossible when the given value of the angle differs more from 90° than that of the leg.

EXAMPLE. Given, in the spherical right triangle (fig. 2.), $a = 35^\circ\ 44'$ and $A = 37^\circ\ 28'$; to solve the triangle.

$$\textit{Ans.}\ \left.\begin{aligned} h &= 73^\circ\ 45', \\ B &= 7[\text{illegible}]^\circ\ 51', \\ b &= 69^\circ\ 50', \end{aligned}\right\}\ \text{or}\ \left\{\begin{aligned} h &= 106^\circ\ 15', \\ B &= 107^\circ\ \ 9', \\ b &= 110^\circ\ 10'. \end{aligned}\right.$$

30. *Problem.* To solve a spherical right triangle, when one of its legs and the adjacent angle are known.

Solution. Let ABC (fig. 2.) be the triangle; a the given leg, and B the given angle.

First. To find the hypothenuse h; co. B is the middle part, and co. h and a are adjacent parts. Hence, by (474),

$$\cos. B = \text{tang. } a \text{ cotan. } h; \tag{584}$$

and, by (7),

$$\text{cotan. } h = \frac{\cos. B}{\text{tang. } a} = \text{cotan. } a \cos. B. \tag{585}$$

Secondly. To find the other angle A; co. A is the middle part, and co. B and a are opposite parts. Hence, by (475),

$$\cos. A = \cos. a \sin. B. \tag{586}$$

Thirdly. To find the other leg b; a is the middle part, and co. B and b are adjacent parts. Hence, by (474),

$$\sin. a = \text{tang. } b \text{ cotan. B}; \tag{587}$$

and, by (7),

$$\text{tang. b} = \frac{\sin. a}{\text{cotan. } B} = \sin. a \text{ tang. } B. \tag{488}$$

Example. Given, in the spherical right triangle, (fig. 2.), $a = 118° 54'$ and $B = 12° 19'$; to solve the triangle.

Ans. $h = 118° 20'$,
$A = 95° 55'$,
$b = 10° 49'$.

31. *Problem.* To solve a spherical right triangle, when its two legs are known.

Solution. Let ABC (fig. 2.) be the triangle, a and b the given legs.

First. To find the hypotheneuse h; co. h is the middle part, a and b are opposite parts. Hence, by (475),

(589) $$\cos. h = \cos. a \cos. b.$$

Secondly. To find one of the angles, as A; b is the middle part, and co. A and a are adjacent parts. Hence, by (474),

(590) $$\sin. b = \text{tang.}\, a \,\text{cotan.}\, A;$$

and, by (7),

(591) $$\text{cotan.}\, A = \frac{\sin. b}{\text{tang.}\, a} = \text{cotan.}\, a \sin. b.$$

In the same way

(592) $$\text{cotan.}\, B = \text{cotan.}\, b \sin. a.$$

Example. Given, in the spherical right triangle (fig. 2.), $a = 1^\circ$ and $b = 100^\circ$; to solve the triangle.

Ans. $h = 100^\circ$,
$A = 1^\circ\ 1'$,
$B = 90^\circ\ 12'$.

32. *Problem.* To solve a spherical right triangle, when the two angles are given.

Solution. Let ABC (fig. 2.) be the triangle, A and B the given angles.

First. To find the hypothenuse h; co. h is the middle part, and co. A and co. B are adjacent parts. Hence, by (474),

$$\cos. h = \text{cotan. } A \text{ cotan. } B. \tag{593}$$

Secondly. To find one of the legs, as a; co. A is the middle part, and co. B and a are the opposite parts. Hence, by (475),

$$\cos. A = \cos. a \sin. B; \tag{594}$$

and, by (7),

$$\cos. a = \frac{\cos. A}{\sin. B} = \cos. A \text{ cosec. } B. \tag{595}$$

In the same way

$$\cos b = \text{cosec. } A \cos. B. \tag{596}$$

33. *Scholium.* The problem is, by (540), impossible when the sum of the given values of A and B (597) is less than 90°, or greater than 270°, or when their difference is greater than 90°.

Example. Given, in the spherical right triangle (fig. 2.),

$$A = 91° \ 11' \text{ and } B = 111° \ 11';$$

to solve the triangle.

Ans. $h = 89° \ 33'$,
$a = 91° \ 16'$,
$b = 111° \ 11'$.

CHAPTER III.

Spherical Oblique Triangles.

SECTION I.

Theorems for the Solution of Spherical Oblique Triangles.

34. *Theorem.* *The sines of the sides in any* (598) *spherical triangle are proportional to the sines of the opposite angles.*

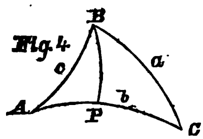


Fig. 4

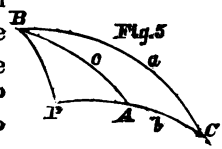


Fig. 5

Demonstration. Let ABC (figs. 4. and 5.) be the given triangle. Denote by a, b, c, the sides respectively opposite to the angles A, B, C. From either of the vertices let fall the perpendicular BP upon the opposite side AC. Then, in the right triangle ABP, making BP the middle part, co. c and co. BAP are the opposite parts. Hence, by (475),

(599) $$\sin. BP = \sin. c \sin. BAP = \sin. c \sin. A.$$

For BAP is either the same as A, or it is its supplement, and in either case has the same sine, by (195).

Again, in triangle BPC, making BP the middle part, co. a and co. C are the opposite parts. Hence, by (475),

(600) $$\sin. BP = \sin. a \sin. C;$$

and, from (599) and (600),

(601) $$\sin. c \sin. A = \sin. a \sin. C,$$

which may be written as a proportion, as follows;

(602) $$\sin. a : \sin. A :: \sin. c : \sin. C.$$

In the same way

$$\sin. a : \sin. A :: \sin. b : \sin. B. \quad (603)$$

35. *Theorem. Bowditch's Rules for Oblique Triangles.* If, in a spherical triangle, two right triangles are formed by a perpendicular let fall from one of its verticles upon the opposite side; and if, in the two right triangles, the middle parts are so taken that the perpendicular is an adjacent part in both of them; then

The sines of the middle parts in the two triangles are proportional to the tangents of the adjacent parts. (604)

But, if the perpendicular is an opposite part in both the triangles, then

The sines of the middle parts are proportional to the cosines of the opposite parts. (605)

Demonstration. Let M denote the middle part in one of the right triangles, A an adjacent part, and O an opposite part. Also let m denote the middle part in the other right triangle, a an adjacent part, and o an opposite part; and let p denote the perpendicular. (606)

First. If the perpendicular is an adjacent part in both triangles, we have, by (474),

$$\sin. M = \text{tang. } A \text{ tang } p, \quad (607)$$

$$\sin. m = \text{tang. } a \text{ tang. } p. \quad (608)$$

The quotient of (607), divided by (608), is

$$\frac{\sin. M}{\sin. m} = \frac{\text{tang. } A \text{ tang. } p}{\text{tang. } a \text{ tang. } p} = \frac{\text{tang. } A}{\text{tang. } a}, \quad (609)$$

or

$$\sin. M : \sin. m :: \text{tang. } A : \text{tang. } a. \quad (610)$$

Secondly. If the perpendicular is an opposite part in both the triangles, we have, by (475),

(611) $$\sin. M = \cos. O \cos. p,$$

(612) $$\sin. m = \cos. o \cos. p.$$

The quotient of (611) divided by (612) is

(613) $$\frac{\sin. M}{\sin. m} = \frac{\cos. O \cos. p}{\cos. o \cos. p} = \frac{\cos. O}{\cos. o},$$

or

(614) $$\sin. M : \sin. m :: \cos O : \cos. o.$$

SECTION. II.

Solution of Spherical Oblique Triangles.

36. *Problem.* To solve a spherical triangle when two of its sides and the included angle are known.

Solution. Let ABC (figs. 4. and 5.) be the triangle; a and b the given sides, and C the given angle. From B let fall on AC the perpendicular BP.

First. To find PC, we know, in the right triangle BPC, the hypothenuse a and the angle C. Hence, by means of (474),

(615) $$\text{tang. } PC = \cos. C \text{ tang. } a.$$

Secondly. AP is the difference between AC and PC, that is,

(616) (fig. 4.) $AP = b - PC$, or (fig. 5.) $AP = PC - b$.

Thirdly. To find the side c. If, in the triangle BPC, co. a is the middle part, PC and PB are

opposite parts; and if, in the triangle ABP, co. c is the middle part, BP and AP are the opposite parts. Hence, by (605),

$$\left\{\begin{array}{l} \cos. PC : \cos. AP :: \sin. (\text{co. } a) : \sin. (\text{co. } c), \\ \text{or} \\ \cos. PC : \cos. AP :: \cos. a : \cos. c. \end{array}\right. \quad (617)$$

Fourthly. To find the angle A. If, in the triangle BPC, PC is the middle part, co. C and BP are adjacent parts; and if, in the triangle ABP, AP is the middle part, co. BAP and BP are adjacent parts. Hence, by (604),

$$\sin. PC : \sin. PA :: \text{cotan. } C : \text{cotan. } BAP, \quad (618)$$

and BAP is the angle A (fig. 4.), when the perpen- (619)
dicular falls within the triangle; or it is the supplement of A (fig. 5.), when the perpendicular falls without the triangle.

Fifthly. B is found by means of (598)

$$\sin. c : \sin. C :: \sin. b : \sin. B. \quad (620)$$

37. *Scholium.* In determining PC, c and BAP,
by (615), (617), and (618), the signs of the several (621)
terms must be carefully attended to; by means of (496).

But to determine which value of B, determined by
(620), is the true value, regard must be had to the (622)
following rules which will be demonstrated hereafter.

I. The greater side of a spherical triangle is always (623)
opposite to the greater angle (768).

(624) II. Each side is less than the sum of the other two (730).

(625) III. The sum of the sides is less than 360° (713).

(626) IV. Each angle is less than the difference between 180°, and the sum of the other two angles (822).

There are, however, cases in which these conditions are all satisfied by each of the values of
(627) B. In any such case this angle can be determined in the same way, in which the angle A was determined by letting fall a perpendicular, from the vertex A on the side BC. But this difficulty can always,
(628) by (772), be avoided by letting fall the perpendicular upon that of the two given sides which differs the most from 90°.

EXAMPLES.

1. Given, in the spherical triangle ABC,

$$a = 45° \; 54', \; b = 138° \; 32', \text{ and } C = 98° \; 44';$$

to solve the triangle.

Solution. By (615),

$C =$	98° 44'.	cos.	9.18137 *n.*
$a =$	45° 54'.	tang.	0.01365
$PC =$	171° 6'.	tang.	9.19502 *n.*

By (616),

$$AP = 171° \; 6' - 138° \; 32' = 32° \; 34'.$$

By (617),

$PC =$	$171^\circ\ 6'$.	cos. (ar. co.)	10.00526 *n.*
$AP =$	$32^\circ\ 34'$.	cos.	9.92571
$a =$	$45^\circ\ 54'$.	cos.	9.84225
$c =$	$126^\circ\ 23'$.	cos.	9.77322 *n.*

By (618),

$PC =$	$171^\circ\ 6'$.	sin. (ar. co.)	10.81048
$AP =$	$32^\circ\ 34'$.	sin.	9.73101
$C =$	$98^\circ\ 44'$.	cotan.	9.18644 *n.*
$BAP =$	$118^\circ\ 7'$.	cotan.	9.72793 *n.*

By (619),

$$A = 180^\circ - 118^\circ\ 7'. = 61^\circ\ 53'.$$

By (620),

$c =$	$126^\circ\ 23'$.	sin. (ar. co.)	10.09417
$C =$	$98^\circ\ 44'$.	sin.	9.99494
$b =$	$138^\circ\ 32'$.	sin.	9.82098
$B =$	$125^\circ\ 37'$.	sin.	9.91009

Ans. $c = 126^\circ\ 23'$,
$A = 61^\circ\ 53'$,
$B = 125^\circ\ 37'$.

2. Given, in the spherical triangle ABC, $a = 100^\circ$, $b = 125$, and $C = 45^\circ$; to solve the triangle.

Ans. $c = 47^\circ\ 55'$,
$A = 69^\circ\ 44'$,
$B = 128^\circ\ 42'$.

38. *Problem.* To solve a spherical triangle, when one of its sides and the two adjacent angles are given.

Solution. Let ABC (figs. 4. and 5.) be the triangle; a the given side, and B and C the given angles.

From B let fall on AC the perpendicular BP.

First. To find PBC, we know, in the right triangle BPC, the hypothenuse a and the angle C. Hence, by (474),

(629) $$\text{cotan.}\ PBC = \cos.\ a \ \text{tang.}\ C.$$

Secondly. ABP is the difference between ABC and PBC, that is,

$$\text{(fig. 4.)} \quad ABP = B - PBC,$$

(630) or

$$\text{(fig. 5.)} \quad ABP = PBC - B.$$

Thirdly. To find the angle A. If, in the triangle PBC, co. C is the middle part, PB and co. PBC are the opposite parts; and if, in the triangle ABP, co. BAP is the middle part, PB and co. ABP are the opposite parts. Hence, by (605),

(631) $$\left\{\begin{array}{l} \cos.\ (\text{co.}\ PBC) : \cos.\ (\text{co.}\ ABP) : : \sin.\ (\text{co.}\ C) : \\ \qquad \sin.\ (\text{co.}\ BAP), \\ \text{or} \\ \sin.\ PBC : \sin.\ ABP : : \cos.\ C : \cos.\ BAP; \end{array}\right.$$

(632) and BAP is either the angle A or its supplement, as in (619).

Fourthly. To find the side c. If, in the triangle PBC, co. PBC is the middle part, PB and co. a

are the adjacent parts; and if, in the triangle ABP, co. ABP is the middle part, PB and co. c are the adjacent parts. Hence, by (604),

$$\cos. PBC : \cos. ABP :: \text{cotan. } a : \text{cotan. } c. \qquad (633)$$

Fifthly. b is found by (598),

$$\sin. C : \sin. c :: \sin. B : \sin. b. \qquad (634)$$

39. *Scholium.* In determining PBC, BAP, and (635)
c by (629), (631), and (633), the signs of the several terms must be carefully attended to, by means of (496).

To determine which value of b, obtained from
(634), is the true value, regard must be had to (623– (636)
626). But if all these conditions are satisfied by both values of b, then b may be calculated by letting fall a perpendicular from C on the side c in the same way in which c has been obtained in the preceding solution. But this case can, by (772), be avoided by letting fall the perpendicular from the vertex of that
one of the two given angles which differs the most (637)
from 90°.

EXAMPLES.

1. Given in the spherical triangle ABC, $a = 175°\ 27'$, $B = 126°\ 12'$, and $C = 109°\ 16'$; to solve the triangle.

Solution. By (629),

$a = 175° 27'$.	cos.	9.99863 n.
$C = 109° 16'$.	tang.	0.45650 n.
$PBC = 19° 19'$.	cotan.	0.45513.

By (630),

$$ABP = 126° 12' - 19° 19' = 106° 53'.$$

By (631),

$PBC = 19° 19'$.	sin. (ar. co.)	10.48045
$ABP = 106° 53'$.	sin.	9.98087
$C = 109° 16'$.	cos.	9.51847 n.
$BAP = 162° 39'$.	cos.	9.97979 n.

$PBC = 19° 19'$.	cos. (ar. co.)	10.02515
$ABP = 106° 53'$.	cos.	9.46303 n.
$a = 175° 27'$.	cotan.	1.09920 n.
$c = 14° 30'$.	cotan.	0.58738

By (632),

$$A = BAP = 162° 39'.$$

By (634),

$C = 109° 16'$.	sin. (ar. co.)	10.02503
$c = 14° 30'$.	sin.	9.39860
$B = 126° 12'$.	sin.	9.90685
$b = 167° 38'$.	sin.	9.33048

Ans. $A = 162° 39'$,
$b = 167° 38'$,
$c = 14° 30'$.

Examples

2. Given, in the spherical triangle ABC,
$a = 45° \, 54'$, $B = 125° \, 37'$, and $C = 98° \, 44'$;
to solve the triangle.

Ans. $A = 61° \, 55'$,
$b = 138° \, 34'$,
$c = 126° \, 26'$.

40. *Problem.* To solve a spherical triangle when two sides and an angle opposite one of them are given.

Solution. Let ABC (figs. 4. and 5.) be the triangle; a and c the given sides, and C the given angle.

From B let fall on AC the perpendicular BP.

First. To find PC. We know, in the right triangle PBC, the side a and the angle C. Hence, by (474),

$$\text{tang. } PC = \cos. \ C \text{ tang. } a. \tag{638}$$

Secondly. To find AP. If, in the triangle PBC, co. a is the middle part, CP and PB are the opposite parts; and, if, in the triangle ABP, co. c is the middle part, AP and PB are the opposite parts. Hence, by (605),

$$\cos. \ a : \cos. \ c : : \cos. \ PC : \cos. \ AP. \tag{639}$$

Thirdly. To find b. There are, in general, two triangles which resolve the problem, in one of which (fig. 4.)

$$b = PC + AP, \tag{640}$$

and in the other (fig. 5.)

$$b = PC - AP. \tag{641}$$

But, if AP is greater than PC, there is but one triangle, as in (fig. 4.), and b is obtained by (640); or,
(642) if the sum of AP and PC is greater than 180°, there is but one triangle, as in (fig. 5.), and b is obtained by (641).

Fourthly. A and B are found by (598).

(643) $$\sin. c : \sin. C :: \sin. a : \sin. A$$

(644) $$\sin. c : \sin. C :: \sin. b : \sin. B$$

41. *Scholium.* In determining PC and AP by
(645) (638) and (639), the signs of the several terms must be carefully attended to by means of (496).

The two values of A, given by (643), correspond respectively to the two triangles which satisfy the problem. And the one, which belongs to each trian-
(646) gle, is to be selected, so that the angle BAP, which is the same as A in (fig. 4.) and the supplement of A in
(647) (fig. 5.), may be obtuse if C is obtuse, and acute if C
(648) is acute. For BP is the side opposite BAP in the right triangle ABP, and the side opposite C in the triangle BCP; and therefore, by (517), BP, BAP, and C are all at the same time less than 90°, or all greater than 90°.

Of the two values of B, given by (644), the one
(649) which belongs to each triangle is to be determined by means of (623 – 626).

42. *Scholium.* The problem is, by (772), impossible, when the given value of c differs more from 90° than that of a; if, at the same time, the value of one
(650) of the two quantities, c and C, is greater than 90° while that of the other is less than 90°. And in this

case we should find that AP was larger than PC, and at the same time that the sum of AP and PC was more than 180°.

EXAMPLES.

1. Given, in the spherical triangle ABC,
$$a = 35°,\ c = 142°,\ C = 176°;$$
to solve the triangle.

Solution. By (638),

$C = 176°$.	cos.	9.99894 n.
$a = 35°$.	tang.	9.84523
$PC = 145° 4'$.	tang.	9.84417 n.

By (639),

$a = 35°$.	cos. (ar. co.)	10.08664
$PC = 145° 4'$.	cos.	9.91372 n.
$c = 142°$.	cos.	9.89653 n.
$AP = 37° 56'$.	cos.	9.89689

By (641),
$$b = 145° 4' - 37° 56' = 107° 8'.$$

By (643),

$c = 142°$.	sin. (ar. co.)	10.21066
$C = 176°$.	sin.	8.84358
$a = 35°$.	sin.	9.75859
$A = 3° 44'$.	sin.	8.81283

By (644),

$c = 142°.$	sin.	(ar. co.)	10.21066
$C = 176°.$	sin.		8.84358
$b = 107°\ 8'.$	sin.		9.98029
$B = 6°\ 13'.$	sin.		9.03453

Ans. $b = 107°\ 8'$,
$A = 3°\ 44'$,
$B = 6°\ 13'$.

2. Given, in the spherical triangle ABC,

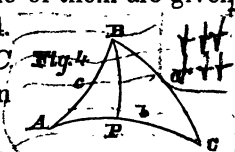
$$a = 54°,\ c = 22°,\ C = 12°;$$

to solve the triangle.

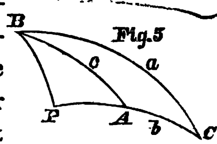
Ans. $\left.\begin{aligned} b &= 73°\ 16', \\ A &= 26°\ 41', \\ B &= 147°\ 53', \end{aligned}\right\}$ or $\left\{\begin{aligned} b &= 33°\ 32', \\ A &= 153°\ 19', \\ B &= 17°\ 51'. \end{aligned}\right.$

43. *Problem.* To solve a spherical triangle when two angles and a side opposite one of them are given.

Solution. Let ABC (figs. 4. and 5.) be the triangle; A and C the given angles, and a the given side.



From B let fall on AC the per-
(651) pendicular BP. This perpendicular will, by (647), fall within the triangle, if A and C are either both obtuse or both acute; but it will fall without if one is obtuse and the other acute.



First. PC may be found, as in (638),

(652) $$\text{tang. } PC = \cos. C \text{ tang. } a.$$

Secondly. To find AP. If, in the triangle PBC, PC is the middle part, co. C and PB are the adjacent parts; and, if, in the triangle ABP, AP is the middle part, co. BAP and BP are the adjacent parts. Hence, by (604),

$$\text{cotan. } C : \text{cotan. } BAP :: \text{sin. } PC : \text{sin. } AP. \qquad (653)$$

Thirdly. To find b. We have

$$\begin{cases} \text{(fig. 4.)} & b = PC + AP, \\ \text{(fig. 5.)} & b = PC - AP. \end{cases} \qquad (654)$$

Fourthly. c and B are found by (538).

$$\text{sin. } A : \text{sin. } a :: \text{sin. } C : \text{sin. } c; \qquad (655)$$

$$\text{sin. } a : \text{sin. } A :: \text{sin. } b : \text{sin.} B. \qquad (656)$$

44. *Scholium.* In determining PC by (652), the signs of the several terms must be attended to by means of (496). (657)

Either value of AP, given by (653), may be used, and there will be two different triangles solving the problem, except when $AP + PC$ (fig. 4.) is greater than 180°, or PC (fig. 5.) is less than AP. It may (658) be that both values of AP satisfy the conditions of the problem, or that only one value satisfies them, or (659) that neither value does; in which last case the problem is impossible.

Of the values of c, determined by (655), the true value must be ascertained from the right triangle ABP, by (495) and (517); or since, as in (648), PB (660) and C are both greater than 90° or both less than 90° at the same time; it follows, from (495), that when C and AP are both greater or both less than 90°, that c is less than 90°; but when one of them is (661) greater and the other less than 90°, c is greater than 90°.

(662) From the two values of B (656) the true value must be selected by means of (623 – 627).

45. *Scholium.* The problem is impossible, by (772), when A differs more from 90° than does C, and when at the same time, one of the two quantities a and A is
(663) less than 90°, while the other is greater than 90°. But this case is precisely the same as the impossible case of (659).

EXAMPLES.

1. Given, in the spherical triangle ABC,

$$A = 95°, \; C = 104°, \text{ and } a = 138°;$$

to solve the triangle.

Solution. By (652),

$C =$	104°.	cos.	9.38368 *n.*
$a =$	138°.	tang.	9.95444 *n.*
$PC =$	12° 17′.	tang.	9.33812

By (653),

$C =$	104°.	cotan. (ar. co.)	0.60323 *n.*
$PC =$	12° 17′.	sin.	9.32786
$BAP =$	95°.	cotan.	8.94195 *n.*
$AP =$	4° 17′.	sin.	8.87304

By (654),

$$b = 12° \, 17' + 4° \, 17' = 16° \, 34'.$$

By (655),

$A = \ \ 95^\circ.$	sin. (ar. co.)	10.00166
$a = 138^\circ.$	sin.	9.82551
$C = 104^\circ.$	sin.	9.98690
$c = 139^\circ\ 20'.$	sin.	9.81407

By (656),

$a = 138^\circ.$	sin. (ar. co.)	10.17449
$A = \ \ 95^\circ.$	sin.	9.99834
$b = \ \ 16^\circ\ 34'.$	sin.	9.45504
$B = \ \ 25^\circ\ \ 7'.$	sin.	9.62787

$$Ans.\quad \begin{aligned} b &= \ \ 16^\circ\ 34', \\ c &= 139^\circ\ 20', \\ B &= \ \ 25^\circ\ \ 7'. \end{aligned}$$

2. Given, in the spherical triangle ABC,

$$A = 104^\circ,\ C = 95^\circ, \text{ and } a = 138^\circ;$$

to solve the triangle.

$$Ans. \left\{ \begin{aligned} b &= \ \ 17^\circ\ 21', \\ c &= 136^\circ\ 36', \\ B &= \ \ 25^\circ\ 37', \end{aligned} \right. \text{ or } \left\{ \begin{aligned} b &= 171^\circ\ 37', \\ c &= \ \ 43^\circ\ 24', \\ B &= 167^\circ\ 47'. \end{aligned} \right.$$

46. *Problem.* To solve a spherical triangle when its three sides are given.

Solution. Let ABC (figs. 4. and 5.) be the triangle; a, b, and c being the given sides.

From B let fall on AC the perpendicular BP.

Then, in the right triangle PBC, if co. C is the middle part, co. a and PC are the adjacent parts. Hence, by (474),

$$\text{cos. } C = \text{cotan. } a \text{ tang. } PC. \qquad (664)$$

If, in the triangle BPC, co. a is the middle part, BP and PC are the opposite parts; and, if, in the triangle ABP, co. c is the middle part, BP and AP are the opposite parts. Hence, by (605),

(665) $$\cos. a : \cos. c :: \cos. PC : \cos. AP.$$

But

(666) (fig. 4.) $AP = b - PC$, and (fig. 5.) $AP = PC - b$.

Hence, by (116) and (202),

(667) $$\cos. AP = \cos. (b - PC) = \cos. (PC - b),$$

(668) $$= \cos. b \cos. PC + \sin. b \sin. PC;$$

which, substituted in (665), gives

(669) $$\cos. a : \cos. c :: \cos. PC : \cos. b \cos. PC + \sin. b \sin. PC.$$

Dividing the two terms of the last ratio of this proportion by cos. PC, and reducing by (10), we have (10)

(670) $$\cos. a : \cos. c :: 1 : \cos. b + \sin. b \text{ tang. } PC.$$

Make the product of the means equal that of the extremes, and we have (PC)

(671) $$\cos. a \cos. b + \cos. a \sin. b \text{ tang. } PC = \cos. c;$$

by transposition

(672) $$\cos. a \sin. b \text{ tang. } PC = \cos. c - \cos. a \cos. b.$$

Divide by sin. a sin. b and reduce the first member by (11)

(673) $$\text{cotan. } a \text{ tang. } PC = \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin b};$$

which, substituted in (664), gives

(674) $$\cos. C = \frac{\cos. c - \cos. a \cos. b}{\sin. a \sin. b},$$

whence the value of the angle C may be calculated, and in the same way either of the other angles.

47. *Corollary.* The equation (674) may be brought into a form more easy for calculation by logarithms,

as follows. Add unity to both its members and it becomes

$$1 + \cos. C = \frac{\cos. c - \cos. a \cos. b + \sin. a \sin. b}{\sin. a \sin. b}. \quad (675)$$

But, by (104),

$$\cos. (a + b) = \cos. a \cos. b - \sin. a \sin. b, \quad (676)$$

which, substituted in the numerator of (675), gives

$$1 + \cos. C = \frac{\cos. c - \cos. (a + b)}{\sin. a \sin. b}. \quad (677)$$

Now we have (122)

$$\cos. (M - N) - \cos. (M + N) = 2 \sin. M \sin. N; \quad (678)$$

and letting s denote half the sum of the sides or

$$s = \tfrac{1}{2} (a + b + c); \quad (679)$$

if we make in (678)

$$\begin{cases} M = \tfrac{1}{2} (a + b + c) = s, \\ N = \tfrac{1}{2} (a + b - c) = s - c; \end{cases} \quad (680)$$

we have

$$\begin{cases} M + N = a + b, \\ M - N = c; \end{cases} \quad (681)$$

and (678) becomes

$$\cos. c - \cos. (a + b) = 2 \sin. s \sin. (s - c); \quad (682)$$

which, substituted in (677), gives

$$1 + \cos. C = \frac{2 \sin. s \sin. (s - c)}{\sin. a \sin. b}. \quad (683)$$

But, by (140),

$$1 + \cos. C = 2 (\cos. \tfrac{1}{2} C)^2, \quad (684)$$

whence

$$2 (\cos. \tfrac{1}{2} C)^2 = \frac{2 \sin. s \sin. (s - c)}{\sin. a \sin. b}, \quad (685)$$

and

$$\cos. \tfrac{1}{2} C = \sqrt{\frac{\sin. s \sin. (s - c)}{\sin. a \sin. b}}. \quad (686)$$

48. *Corollary.* The angles A and B may be found by the two following equations which are easily deduced from (686),

(687) $$\cos.\ \tfrac{1}{2} A = \sqrt{\frac{\sin.\ s \sin.\ (s - a)}{\sin.\ b \sin.\ c}}$$

(688) $$\cos.\ \tfrac{1}{2} B = \sqrt{\frac{\sin.\ s \sin.\ (s - b)}{\sin.\ a \sin.\ c}}$$

49. *Corollary.* The equation (674) can be brought into another form equally simple in calculation. Subtract each member from unity

(689) $$1 - \cos.\ C = \frac{\sin.\ a \sin b - \cos.\ c + \cos.\ a \cos b}{\sin.\ a \sin.\ b}.$$

But, by (116),

(690) $$\cos.\ (a - b) = \cos.\ a \cos.\ b + \sin.\ a \sin.\ b,$$

which substituted in the numerator of (689) gives

(691) $$1 - \cos.\ C = \frac{\cos.\ (a - b) - \cos.\ c}{\sin.\ a \sin.\ b.}.$$

Now, if in (678),

(692) $$\cos.\ (M - N) - \cos.\ (M + N) = 2 \sin.\ M \sin.\ N;$$

we make

(693) $$\begin{cases} M = \tfrac{1}{2}(a - b + c) = s - b, \\ N = \tfrac{1}{2}(-a + b + c) = s - a; \end{cases}$$

we have

(694) $$\begin{cases} M + N = c, \\ M - N = a - b; \end{cases}$$

and (692) becomes

(695) $$\cos.\ (a - b) - \cos.\ c = 2 \sin.\ (s - a) \sin.\ (s - b);$$

which, substituted in (691), gives

(696) $$1 - \cos.\ C = \frac{2 \sin.\ (s - a) \sin.\ (s - b)}{\sin.\ a \sin.\ b}.$$

But, by (141),

(697) $$1 - \cos.\ C = 2 (\sin.\ \tfrac{1}{2} C)^2;$$

whence

$$2\,(\sin.\tfrac{1}{2}\,C)^2 = \frac{2\sin.(s-a)\sin.(s-b)}{\sin.a\sin.b}, \tag{698}$$

and

$$\sin.\tfrac{1}{2}\,C = \sqrt{\frac{\sin.(s-a)\sin.(s-b)}{\sin.a\sin.b}}. \tag{699}$$

50. *Corollary.* In the same way we might deduce the following equations,

$$\sin.\tfrac{1}{2}\,A = \sqrt{\frac{\sin.(s-b)\sin.(s-c)}{\sin.b\sin.c}} \tag{700}$$

$$\sin.\tfrac{1}{2}\,B = \sqrt{\frac{\sin.(s-a)\sin.(s-c)}{\sin.a\sin.c}} \tag{701}$$

51. *Corollary.* The quotient of (700) divided by (687) is by (10)

$$\text{tang.}\tfrac{1}{2}\,A = \frac{\sin.\frac{1}{2}\,A}{\cos.\frac{1}{2}\,A} = \sqrt{\frac{\sin.(s-b)\sin.(s-c)}{\sin.s\sin.(s-a)}}. \tag{702}$$

In the same way

$$\text{tang.}\tfrac{1}{2}\,B = \sqrt{\frac{\sin.(s-a)\sin.(s-c)}{\sin.s\sin.(s-b)}} \tag{703}$$

$$\text{tang.}\tfrac{1}{2}\,C = \sqrt{\frac{\sin.(s-a)\sin.(s-b)}{\sin.s\sin.(s-c)}} \tag{704}$$

EXAMPLES.

1. Given, in the spherical triangle ABC,

$$a = 46^\circ,\ b = 72^\circ,\ \text{and}\ c = 68^\circ;$$

to solve the triangle.

Solution.	By (615),	by (616),	by (614),
$a=46°$ sin.		(ar.co.) 10.14307	(ar.co.) 10.14307
$b=72°$ sin.	(ar.co.) 10.02179		(ar.co.) 10.02179
$c=68°$ sin.	(ar.co.) 10.03283	(ar.co.) 10.03283	
$s=93°$ sin.	9.99940	9.99940	9.99940
$s-a=47°$ sin.	9.86413		
$s-b=21°$ sin.		9.55433	
$s-c=25°$ sin.			9.62595
	2)19.91815	2)19.72963	2)19.79021
cos.	9.95908	9.86482	9.89511

$\frac{1}{2} A = 24° \, 29'$, $\frac{1}{2} B = 42° \, 54'$, $\frac{1}{2} C = 38° \, 14'$;

Ans. $A = 48° \, 58'$, $B = 85° \, 48'$, $C = 76° \, 28'$.

2. Given, in the spherical triangle ABC, $a = 3°$, $b = 4°$, $c = 5°$; to solve the triangle.

Ans. $A = 36° \, 55'$,
$B = 53° \, 10'$,
$C = 90° \, 2'$.

52. *Lemma.* The sine, cosine, secant, and cose-
(705) cant of an angle, greater than 180° and less than 270°, are negative; but its tangent and cotangent are positive.

Demonstration. Let the excess of the angle above 180° be M, which must be less than 90°; and the
(706) angle is $180° + M$. Now, if we change $-N$ into M in (189–194), they become, by (196–201),

(707) $\sin. (180° + M) = \sin. (-M) = -\sin. M$,

(708) $\sin. (180° + M) = -\cos. (-M) = -\cos. M$,

(709) $\text{tang.} (180° + M) = -\text{tang.} (-M) = \text{tang.} M$,

(710) $\cot. (180° + M) = -\text{cotan.} (-M) = \text{cotan.} M$,

sec. $(180° + M) = -\text{sec.}\ (-M) = -\text{sec.}\ M,$ (711)
cosec. $(180° + M) = \text{cosec.}\ (-M) = -\text{cosec.}\ M,$ (712)
which agree with (705).

53. *Theorem. The sum of the sides of a spherical triangle is less than* 360°. (713)

Demonstration. We are to prove that $2\,s$ (679) is less than 360°, or that s is less than 180°. (714)

Since sin. b and sin. c are positive, the denominator of the fraction under the radical sign in (687) is positive; and, therefore, the numerator must be likewise positive. (715)

But if s were greater than 180°, sin s would by (705) be negative, since s must be less than 270°, as each side is less than 180°, and consequently the sum of the sides is less than 540°. (716) Now, sin s being negative the other factor of the numerator of (687) sin $(s - a)$, must by (715) likewise be negative; (717) that is by (202), $(s - a)$ must be negative. For it cannot be greater than 180°; since by (693)

$$s - a = \tfrac{1}{2}\,(b + c - a), \qquad (718)$$

and it is therefore less than $\frac{1}{2}\,(b + c)$ or by (716) less than 180°.

But if $(s - a)$ is negative, or

$$s < a\,; \qquad (719)$$

it may be proved in the same way from (688) and (686), that

$$s < b, \text{ and } s < c, \qquad (720)$$

the sum of which is

$$3\,s < a + b + c, \qquad (721)$$

or by (679),

$$3\,s < 2\,s\,; \qquad (722)$$

which is impossible and therefore s is not greater than
(723) $180°$.

(724) Neither is s equal to $180°$, for if it were the expressions (686), (687), and (688) would vanish.

(725) Therefore s must be less than $180°$, as in (714).

54. *Corollary.* Since s is less than $180°$, sin. s
(726) is positive and therefore by (715), sin. $(s - a)$ is positive or $(s - a)$ is positive, that is,

(727) $$s > a, \text{ or } 2\,s > 2\,a,$$

or by (679)

(728) $$a + b + c > 2\,a,$$

or

(729) $$b + c > a\,;$$

that is, *each side of a spherical triangle is less than*
(730) *the sum of the other two.*

55. *Theorem. In an isosceles spherical triangle*
(731) *the angles opposite the eqaal sides are equal.*

Demonstration. If

(732) $$a = b,$$

the expressions for cos. $\frac{1}{2}$ A and cos. $\frac{1}{2}$ B become equal and therefore

(733) $$A = B.$$

56. *Corollary. An equilateral spherical triangle*
(734) *is also equiangular.*

57. Lord Napier obtained two theorems for the solution of a spherical triangle, when a side and the two adjacent angles are given, by which the two sides
(735) can be calculated without the necessity of calculating

the third angle. These theorems, which are given in (749) and (758), can be obtained from equations (702 – 704) by the assistance of the following lemmas.

58. *Lemma.* If we have the equation

$$\frac{\text{tang. } M}{\text{tang. } N} = \frac{x}{y}, \qquad (736)$$

we can deduce from it the following equation,

$$\frac{\sin. (M + N)}{\sin. (M - N)} = \frac{x + y}{x - y}. \qquad (737)$$

Demonstration. We have from (10)

$$\text{tang. } M = \frac{\sin. M}{\cos. M}, \text{ and tang. } N = \frac{\sin. N}{\cos. N}; \qquad (738)$$

which, substituted in (736), give

$$\frac{\sin. M \cos. N}{\cos. M \sin. N} = \frac{x}{y}. \qquad (739)$$

This equation is the same as the proportion

$$\sin. M \cos. N : \cos. M \sin. N : : x : y; \qquad (740)$$

hence, by the theory of proportions,

$$\sin. M \cos. N + \cos. M \sin. N : \sin. M \cos. N - \cos. M \sin. N : : x + y : x - y, \qquad (741)$$

or, by (84) and [illegible],

$$\sin. (M + N) : \sin. (M - N) : : x + y : x - y; \qquad (742)$$

which may be written in the form of an equation as in (737).

59. *Lemma.* If we have the equation

$$\text{tang. } M \text{ tang. } N = \frac{x}{y}; \qquad (743)$$

we can deduce from it the equation

(744) $$\frac{\cos.(M+N)}{\cos.(M-N)} = \frac{y-x}{y+x}.$$

Demonstration. The equations (738) substituted in (743) give

(745) $$\frac{\sin.M\sin.N}{\cos.M\cos.N} = \frac{x}{y}.$$

This equation is the same as the proportion

(746) $$\cos.M\cos.N : \sin.M\sin.N :: y : x;$$

hence, by the theory of proportions,

(747) $$\cos.M\cos.N - \sin.M\sin.N : \cos.M\cos.N + \sin.M\sin.N :: y-x : x+y,$$

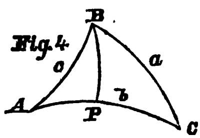
or, by (104) and (116),

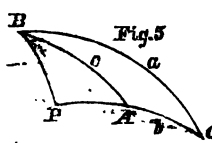
(748) $$\cos.(M+N) : \cos.(M-N) :: y-x : y+x;$$

which may be written as in (744).

60. *Theorem. The sine of half the sum of two angles of a spherical triangle is to the sine of half their difference, as the tangent of half the side to*
(749) *which they are both adjacent is to the tangent of half the difference of the other two sides;* that is, in the spherical triangle ABC (figs. 4. and 5.),

(750) $$\sin.\tfrac{1}{2}(A+C) : \sin.\tfrac{1}{2}(A-C) :: \text{tang.}\tfrac{1}{2}b : \text{tang.}\tfrac{1}{2}(a-c).$$





Demonstration. The quotient of (702), divided by (704) is, by an easy reduction,

$$\frac{\text{tang.}\ \frac{1}{2}A}{\text{tang.}\ \frac{1}{2}C} = \frac{\text{sin.}\ (s-c)}{\text{sin.}\ (s-a)}. \tag{751}$$

Hence, by (736) and (737),

$$\frac{\text{sin.}\ \frac{1}{2}(A+C)}{\text{sin.}\ \frac{1}{2}(A-C)} = \frac{\text{sin.}\ (s-c) + \text{sin.}\ (s-a)}{\text{sin.}\ (s-c) - \text{sin.}\ (s-a)}. \tag{752}$$

But we have by (228), accenting the letters so as not to confound them with the angles of the triangle,

$$\frac{\text{sin.}\ A' + \text{sin.}\ B'}{\text{sin.}\ A' - \text{sin.}\ B'} = \frac{\text{tang.}\ \frac{1}{2}(A'+B')}{\text{tang.}\ \frac{1}{2}(A'-B')}. \tag{753}$$

If we make in this equation

$$\begin{cases} A' = s - c = \frac{1}{2}(a+b-c), \\ B' = s - a = \frac{1}{2}(-a+b+c); \end{cases} \tag{754}$$

we have

$$\begin{cases} A' + B' = b, \\ A' - B' = a - c; \end{cases} \tag{755}$$

and (753) becomes

$$\frac{\text{sin.}\ (s-c) + \text{sin.}\ (s-a)}{\text{sin.}\ (s-c) - \text{sin.}\ (s-a)} = \frac{\text{tang.}\ \frac{1}{2}b}{\text{tang.}\ \frac{1}{2}(a-c)}. \tag{756}$$

This equation, substituted in the second member of (752), gives

$$\frac{\text{sin.}\ \frac{1}{2}(A+C)}{\text{sin.}\ \frac{1}{2}(A-C)} = \frac{\text{tang.}\ \frac{1}{2}b}{\text{tang.}\ \frac{1}{2}(a-c)}; \tag{757}$$

which is the same as (750).

61. *Theorem. The cosine of half the sum of two angles of a spherical triangle is to the cosine of half their difference, as the tangent of half the side to which they are both adjacent is to the tangent of half the sum of the other two sides;* that is, in the spherical triangle ABC (figs. 4. and 5.) (758)

$$\text{cos.}\ \tfrac{1}{2}(A+C) : \text{cos.}\ \tfrac{1}{2}(A-C) :: \text{tang.}\ \tfrac{1}{2}b : \text{tang.}\ \tfrac{1}{2}(a+c). \tag{759}$$

Demonstration. The product of (702) and (704), is, by a simple reduction,

(760) $$\text{tang.}\ \tfrac{1}{2}A\ \text{tang.}\ \tfrac{1}{2}C = \frac{\sin.(s-b)}{\sin. s}.$$

Hence, by (743) and (744),

(761) $$\frac{\cos.\frac{1}{2}(A+C)}{\cos.\frac{1}{2}(A-C)} = \frac{\sin. s - \sin.(s-b)}{\sin. s + \sin.(s-b)}.$$

But (753) is, when inverted,

(762) $$\frac{\sin. A' - \sin. B'}{\sin. A' + \sin. B'} = \frac{\text{tang.}\frac{1}{2}(A'-B')}{\text{tang.}\frac{1}{2}(A'+B')}.$$

If in this equation we make

(763) $$\begin{cases} A' = s = \frac{1}{2}(a+b+c), \\ B' = s - b = \frac{1}{2}(a-b+c); \end{cases}$$

we have

(764) $$\begin{cases} A' + B' = a + c, \\ A' - B' = b; \end{cases}$$

and (762) becomes

(765) $$\frac{\sin. s - \sin.(s-b)}{\sin. s + \sin.(s-b)} = \frac{\text{tang.}\frac{1}{2}b}{\text{tang.}\frac{1}{2}(a+c)}.$$

This equation, substituted in (761), gives

(766) $$\frac{\cos.\frac{1}{2}(A+C)}{\cos.\frac{1}{2}(A-C)} = \frac{\text{tang.}\frac{1}{2}b}{\text{tang.}\frac{1}{2}(a+c)},$$

which is the same as (759).

62. *Scholium.* In using (749) and (758), the signs
(767) of the terms must be attended to by means of (496).

EXAMPLES.

1. Given in the spherical triangle ABC (figs. 4. and 5.)

$$A = 158°, C = 98°, b = 144°;$$

to find sides a and c.

Solution. By (750),

$\frac{1}{2}(A + C) =$	$128°.$	sin. (ar. co.)	10.10347
$\frac{1}{2}(A - C) =$	$30°.$	sin.	9.69897
$\frac{1}{2}b =$	$72°.$	tang.	0.48822
$\frac{1}{2}(a - c) =$	$62° 53'.$	tang.	0.29066

By (759),

$\frac{1}{2}(A + C) =$	$128°.$	cos. (ar. co.)	10.21066 n.
$\frac{1}{2}(A - C) =$	$30°.$	cos.	9.93753
$\frac{1}{2}b =$	$72°.$	tang.	0.48822
$\frac{1}{2}(a + c) =$	$103°.$	tang.	0.63641 n.

Ans. $a = 165° 53'$,
$c = 40° 7'$.

2. Given in the spherical triangle ABC (figs. 4. and 5.)

$$A = 170°, C = 2°, b = 92°;$$

to find a and c.

Ans. $a = 103° 7'$,
$c = 11° 17'$.

63. *Theorem. The greater side of a spherical triangle is opposite the greater angle.* (768)

Demonstration. The first and third terms of the proportion (750),

(769) $$\sin. \tfrac{1}{2}(A + C) : \sin. \tfrac{1}{2}(A - C) : : \text{tang.} \tfrac{1}{2} b : \text{tang.} \tfrac{1}{2}(a - c),$$

are, by (496), both positive, since $\frac{1}{2}(A + C)$ is less
(770) than 180° and $\frac{1}{2} b$ is less than 90°. The second and fourth terms must then be both positive or both negative at the same time. But as $\frac{1}{2}(A - C)$ and
(771) $\frac{1}{2}(a - c)$ are both less than 90°, these terms can be negative only when A is less than C and a less than c which agrees with (768).

64. *Theorem. Of two sides of a spherical triangle the one which differs most from* 90° *must be*
(772) *opposite the angle which differs most from* 90°; *and this side and angle must be both greater or both less than* 90°.

Demonstration. Let the side a differ more from
(773) 90°, than does the side b; then by (507) sin. a is less than sin. b. But, by (598),

(774) $$\sin. a : \sin. b : : \sin. A : \sin. B.$$

Hence, sin. A is less than sin. B, and by (507) the
(775) angle A differs more from 90° than does the angle B which agrees with the first part of (772).

Again, if a is also greater than b it must be greater than 90°; and the opposite angle A must, by (768),
(776) be greater than the angle B, and, by (775), differing more from 90° must also be greater than 90°. But if a is less than b, it must, by (773), be acute; and A
(777) must, by (768), be less than B, and, by (775), it must also be acute; which is the second part of (772).

65. *Problem.* To solve a spherical triangle when its three angles are given.

Solution. Let ABC (figs. 4. and 5.) be the triangle, the angles A, B, and C being given.

From B let fall on AC the perpendicular BP.

Then, if, in the right triangle PBC, co. a is made the middle part, co. C and co. PBC are the adjacent parts. Hence, by (474),

$$\cos. a = \text{cotan. } C \text{ cotan. } PBC. \tag{778}$$

If, in the right triangle PBC, co. C is the middle part, co. PBC and PB are the opposite parts; and, if, in the triangle ABP, co. BAP is the middle part, co. PBA and PB are the opposite parts. Hence, by (605),

$$\cos. C : \cos. BAP :: \sin. PBC : \sin. PBA. \tag{779}$$

But

$$\text{(fig. 4.) } BAP = A, \text{ and (fig. 5.) } BAP = 180^\circ - A; \tag{780}$$

also,

$$\text{(fig. 4.)} \quad PBA = B - PBC,$$

and (781)

$$\text{(fig. 5.)} \quad PBA = PBC - B.$$

Hence, and by (190),

$$\begin{aligned} &\text{(fig. 4.)} \quad \cos. BAP = \cos. A, \\ &\text{(fig. 5.)} \quad \cos. BAP = -\cos. A; \end{aligned} \tag{782}$$

also by (781), (91), and (202),

$$\begin{aligned} \text{(fig. 4.)} \quad \sin. PBA &= \sin. (B - PBC) \\ &= \sin. B \cos. PBC - \cos. B \sin. PBC, \end{aligned} \tag{783}$$

$$\begin{aligned} \text{(fig. 5.)} \quad \sin. PBA &= \sin. (PBC - B) \\ &= -\sin. (B - PBC) \\ &= -\sin. B \cos. PBC + \cos. B \sin. PBC; \end{aligned} \tag{784}$$

whence (779) becomes (fig. 4.)

$$\begin{aligned} &\cos. C : \cos. A :: \sin. PBC \\ &: \sin. B \cos. PBC - \cos. B \sin. PBC; \end{aligned} \tag{785}$$

and, (fig. 5.)

(786) $$\cos. C : -\cos. A :: \sin. PBC : -\sin. B \cos PBC + \cos. B \sin. PBC,$$

which becomes the same as (785) by changing the signs of the second and fourth terms.

Divide the two terms of the second ratio of (785) by sin. PBC and reduce, by (11),

(787) $$\cos. C : \cos. A :: 1 : \sin. B \operatorname{cotan}. PBC - \cos. B.$$

Make the product of the means equal that of the extremes, and we have

(788) $$\sin. B \cos. C \operatorname{cotan}. PBC - \cos. B \cos. C = \cos. A;$$

by transposition,

(789) $$\sin. B \cos. C \operatorname{cotan}. PBC = \cos. A + \cos. B \cos. C.$$

Divide by sin. B sin. C, and reduce by (11)

(790) $$\operatorname{cotan}. C \operatorname{cotan}. PBC = \frac{\cos. A + \cos. B \cos. C}{\sin. B \sin. C},$$

which, subtituted in (778), gives

(791) $$\cos. a = \frac{\cos. A + \cos. B \cos. C}{\sin. B \sin. C},$$

whence the value of the side a may be calculated, and in the same way either of the other sides.

66. *Corollary.* The equation (791) may be brought into a form more easy for calculation by logarithms, as follows.

Subtract each member from unity and it becomes

(792) $$1 - \cos. a = \frac{\sin. B \sin. C - \cos. A - \cos. B \cos. C}{\sin. B \sin. C}.$$

But, by (104), changing the signs

(793) $$-\cos. (B + C) = -\cos. B \cos. C + \sin. B \sin. C;$$

which, substituted in the numerator of (792), gives

(794) $$1 - \cos. a = \frac{-\cos. (B + C) - \cos. A}{\sin. B \sin. c}.$$

Now we have by (121), changing the signs,

$$- \cos. (M + N) - \cos. (M - N) \\ = - 2 \cos. M \cos. N; \tag{795}$$

and letting S denote half the sum of the angles or

$$S = \tfrac{1}{2} (A + B + C); \tag{796}$$

if we make in (795),

$$\begin{cases} M = \tfrac{1}{2} (A + B + C) = S, \\ N = \tfrac{1}{2} (- A + B + C) = S - A; \end{cases} \tag{797}$$

we have

$$\begin{cases} M + N = B + C, \\ M - N = A; \end{cases} \tag{798}$$

and (795) becomes

$$- \cos. (B + C) - \cos. A \\ = - 2 \cos. S \cos. (S - A); \tag{799}$$

which, substituted in (794), gives

$$1 - \cos. a = \frac{- 2 \cos. S \cos. (S - A)}{\sin. B \sin. C}. \tag{800}$$

But, by (141), we have

$$1 - \cos. a = 2 (\sin. \tfrac{1}{2} a)^2; \tag{801}$$

whence,

$$2 (\sin. \tfrac{1}{2} a)^2 = \frac{- 2 \cos. S \cos. (S - A)}{\sin. B \sin. C}, \tag{802}$$

and

$$\sin. \tfrac{1}{2} a = \sqrt{\frac{- \cos. S \cos. (S - A)}{\sin. B \sin. C}}. \tag{803}$$

67. *Corollary.* The sides b and c may be found by the two following equations which are readily deduced from (803),

(804) $$\sin. \tfrac{1}{2} b = \sqrt{\frac{-\cos. S \cos. (S - B)}{\sin. A \sin. C}}.$$

(805) $$\sin. \tfrac{1}{2} c = \sqrt{\frac{-\cos. S \cos. (S - C)}{\sin. A \sin. B}}.$$

68. *Corollary.* The equation (791) can be brought into another form equally simple for calculation. Add each member to unity and it becomes

(806) $$1 + \cos. a = \frac{\cos. A + \cos. B \cos. C + \sin. B \sin. C}{\sin. B \sin. C}.$$

But, by (116),

(807) $$\cos. (B - C) = \cos. B \cos. C + \sin. B \sin. C;$$

which, substituted in (806), gives

(808) $$1 + \cos. a = \frac{\cos. A + \cos. (B - C)}{\sin. B \sin. C}.$$

Now we have, by (121),

(809) $$\cos. (M + N) + \cos. (M - N) = 2 \cos. M \cos. N;$$

and if we make

(810) $$\begin{cases} M = \tfrac{1}{2}(A + B - C) = S - C, \\ N = \tfrac{1}{2}(A - B + C) = S - B; \end{cases}$$

we have

(811) $$\begin{cases} M + N = A, \\ M - N = B - C; \end{cases}$$

and (809) becomes

(812) $$\begin{aligned} &\cos. A + \cos. (B - C) \\ &= 2 \cos. (S - B) \cos. (S - C); \end{aligned}$$

which, substituted in (808), gives

(813) $$1 + \cos. a = \frac{2 \cos. (S - B) \cos. (S - C)}{\sin. B \sin. C}.$$

But, by (140) we have

(814) $$1 + \cos. a = 2 (\cos. \tfrac{1}{2} a)^2;$$

whence

$$2\,(\cos.\ \tfrac{1}{2}\,a)^2 = \frac{2\cos.\ (S-B)\cos.\ (S-C)}{\sin.\ B\sin.\ C}, \tag{815}$$

and

$$\cos.\ \tfrac{1}{2}\,a = \sqrt{\frac{\cos.\ (S-B)\cos.\ (S-C)}{\sin.\ B\sin.\ C}}. \tag{816}$$

69. *Corollary.* In the same way we might deduce the following equations,

$$\cos.\ \tfrac{1}{2}\,b = \sqrt{\frac{\cos.\ (S-A)\cos.\ (S-C)}{\sin.\ A\sin.\ C}}, \tag{817}$$

$$\cos.\ \tfrac{1}{2}\,c = \sqrt{\frac{\cos.\ (S-A)\cos.\ (S-B)}{\sin.\ A\sin.\ B}}. \tag{818}$$

70. *Corollary.* The quotient of (803) divided by (816) is

$$\text{tang.}\ \tfrac{1}{2}\,a = \sqrt{\frac{-\cos.\ S\cos.\ (S-A)}{\cos.\ (S-B)\cos.\ (S-C)}}. \tag{819}$$

In the same way

$$\text{tang.}\ \tfrac{1}{2}\,b = \sqrt{\frac{-\cos.\ S\cos.\ (S-B)}{\cos.\ (S-A)\cos.\ (S-C)}}, \tag{820}$$

$$\text{tang.}\ \tfrac{1}{2}\,c = \sqrt{\frac{-\cos.\ S\cos.\ (S-C)}{\cos.\ (S-A)\cos.\ (S-B)}}. \tag{821}$$

EXAMPLE. Given, in the spherical triangle ABC, (figs 4. and 5.),

$$A = 89^\circ,\ B = 5^\circ,\ C = 88^\circ;$$

to solve the triangle.

Ans. $a = 53^\circ\ 10'$,
$b = 4^\circ$,
$c = 53^\circ\ 8'$.

(822) 71. *Theorem. Each angle of a spherical triangle is greater than the difference between* 180° *and the sum of the other two angles.*

Demonstration. Since sin. B and sin. C are posi-
(823) tive, the denominator of the fraction under the radical sign in (803) is positive; and therefore its numerator must likewise be positive.

Now if S were less than 90°, cos. S would be positive, and (— cos. S) would be negative; and, the other factor of the numerator of (803), cos. $(S - A)$
(824) must, by (823), be negative. $(S - A)$ must, then, by (496) and (202), be greater than 90° or less than (— 90°). But it cannot be greater than 90° while S is less than 90°; neither can it be less than — 90°,
(825) for, by (797),

(826) $S - A = \frac{1}{2}(-B + C - A) = \frac{1}{2}(B + C) - \frac{1}{2}A,$

(827) and $\frac{1}{2}A$ is less than 90° by article 1. S cannot then be less than 90°.

(828) Neither can S be equal to 90°, for, in this case, the expressions (803–808) vanish.

(829) S must then be greater than 90°, or 2 S, *the sum of the angles, must be greater than* 180°, that is, each angle is greater than the excess of 180° over the sum
(830) of the other two angles.

But, since S is greater than 90°, cos S must, by (496), be negative, or (— cos. S) must be positive;
(831) and therefore cos. $(S - A)$ must likewise, by (823), be positive. $(S - A)$ must then be less than 90°, or by (826),

(832)
$$\tfrac{1}{2}(B + C - A) < 90°,$$
or

(833)
$$B + C - A < 180°,$$

or, by transposition,

$$A > B + C - 180°;$$ (834)

that is, each angle is greater than the excess of the sum of the other two over 180°; which result, combined with (830), is the same as (822). (835)

72. *Theorem. If, in a spherical triangle, two angles are equal, the opposite sides are also equal, and the triangle is isosceles.* (836)

Demonstration. For, when

$$A = B,$$ (837)

the expressions for sin. $\frac{1}{2}$ a and sin. $\frac{1}{2}$ b (803) and (804) are identical, and therefore

$$a = b.$$ (838)

73. *Corollary. An equiangular spherical triangle is also equilateral.* (839)

74. There are two theorems similar to (749) and (758), which were given by Lord Napier for the solution of the case in which two sides and the included angle are given. By these theorems (844) and (852) the other two angles can be found without the necessity of calculating the third side. (840)

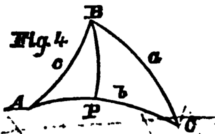
75. *Lemma.* The quotient of (128), divided by (127), is, by (10) and (11), accenting the letters,

$$\frac{\cos. B' - \cos. A'}{\cos. B' + \cos. A'} = \frac{\sin. \frac{1}{2}(A' + B') \sin. \frac{1}{2}(A' - B')}{\cos. \frac{1}{2}(A' + B') \cos. \frac{1}{2}(A' - B')}$$ (841)

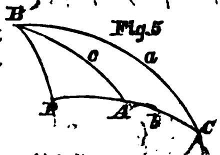
$$= \text{tang.} \tfrac{1}{2}(A' + B') \text{ tang.} \tfrac{1}{2}(A' - B')$$ (842)

$$= \frac{\text{tang.} \frac{1}{2}(A' + B')}{\text{cotan.} \frac{1}{2}(A' - B')} = \frac{\text{tang.} \frac{1}{2}(A' - B')}{\text{cotan.} \frac{1}{2}(A' + B')}$$ (843).

76. *Theorem. The sine of half the sum of two sides of a spherical triangle is to the tangent of half*
(844) *their difference, as the cotangent of half the included angle is to the tangent of half the difference of the other two angles*, that is, in ABC (figs. 4. and 5.),



(845) $\sin. \frac{1}{2}(a+c) : \sin. \frac{1}{2}(a-c)$
$:: \text{cotan.} \frac{1}{2} B : \text{tang.} \frac{1}{2}(A-C)$.



Demonstration. The quotient of (819) divided by (821) is, by simple reduction,

(846) $$\frac{\text{tang.} \frac{1}{2} a}{\text{tang.} \frac{1}{2} b} = \frac{\cos. (S-A)}{\cos. (S-C)}.$$

Hence, by (736) and (737),

(847) $$\frac{\sin. \frac{1}{2}(a+c)}{\sin. \frac{1}{2}(a-c)} = \frac{\cos. (S-A) + \cos. (S-C)}{\cos. (S-A) - \cos. (S-C)}.$$

But if, in (843), we make

(848) $$\begin{cases} A' = S - C = \frac{1}{2}(A+B-C), \\ B' = S - A = \frac{1}{2}(-A+B+C); \end{cases}$$

we have

(849) $$\begin{cases} A' + B' = B, \\ A' - B' = A - C; \end{cases}$$

and (843) becomes, by inverting the fractions,

(850) $$\frac{\cos. (S-A) + \cos. (S-C)}{\cos. (S-A) - \cos. (S-C)} = \frac{\text{cotan.} \frac{1}{2} B}{\text{tang.} \frac{1}{2}(A-C)}.$$

This equation, substituted in (847), gives

(851) $$\frac{\sin. \frac{1}{2}(a+c)}{\sin. \frac{1}{2}(a-c)} = \frac{\text{cotan.} \frac{1}{2} B}{\text{tang.} \frac{1}{2}(A-C)},$$

which is the same as (845).

77. *Theorem. The cosine of half the sum of two sides of a triangle is to the cosine of half their difference, as the cotangent of half the included angle is to the tangent of half the sum of the other two angles*, or in (figs. 4. and 5.), (852)

$$\cos.\ \tfrac{1}{2}\,(a+c) : \cos.\ \tfrac{1}{2}\,(a-c) :: \text{cotan.}\ \tfrac{1}{2}\,B : \text{tang.}\ \tfrac{1}{2}\,(A+C). \quad (853)$$

Demonstration. The product of (819) and (821) is, by a simple reduction,

$$\text{tang.}\ \tfrac{1}{2}\,a\ \text{tang.}\ \tfrac{1}{2}\,C = \frac{-\cos.\ S}{\cos.\ (S-B)}. \quad (854)$$

Hence, by (743) and (744),

$$\frac{\cos.\ \frac{1}{2}\,(a+c)}{\cos.\ \frac{1}{2}\,(a-c)} = \frac{\cos.\ (S-B)+\cos.\ S}{\cos.\ (S-B)-\cos.\ S}. \quad (855)$$

But if in (843) we make

$$\begin{cases} B' = S - B = \tfrac{1}{2}\,(A-B+C), \\ A' = S = \tfrac{1}{2}\,(A+B+C)\,; \end{cases} \quad (856)$$

we have

$$\begin{cases} A' + B' = A + C, \\ A' - B' = B\,; \end{cases} \quad (857)$$

and (843) becomes by inverting the fractions

$$\frac{\cos.\ (S-B)+\cos.\ S}{\cos.\ (S-B)-\cos.\ S} = \frac{\text{cotan.}\ \frac{1}{2}\,B}{\text{tang.}\ \frac{1}{2}\,(A+C)}. \quad (858)$$

This equation being substituted in (855) gives

$$\frac{\cos.\ \frac{1}{2}\,(a+c)}{\cos.\ \frac{1}{2}\,(a-c)} = \frac{\text{cotan.}\ \frac{1}{2}\,B}{\text{tang.}\ \frac{1}{2}\,(A+C)}, \quad (859)$$

which is the same as (853).

78. *Corollary.* In using (844) and (852) regard must be had to the signs of the terms by means of (496). (860)

EXAMPLES.

1. Given in the spherical triangle ABC (figs. 4. and 5.),

$$a = 149°, \ c = 49°, \text{ and } B = 88°;$$

to find the angles A and C.

Solution. By (845),

$\frac{1}{2}(a + c) = 99°.$	sin.	(ar. co.)	10.00538
$\frac{1}{2}(a - c) = 50°.$	sin.		9.88425
$\frac{1}{2}B = 44°.$	cotan.		0.01516
$\frac{1}{2}(A - C) = 38° \ 46'.$	tang.		9.90479

By (853),

$\frac{1}{2}(a + c) = 99°.$	cos.	(ar. co.)	10.80567 *n.*
$\frac{1}{2}(a - c) = 50°.$	cos.		9.80807
$\frac{1}{2}B = 44°.$	cotan.		0.01516
$\frac{1}{2}(A + C) = 103° \ 14'.$	tang.		0.62890 *n.*

Ans. $A = 142°,$
$C = 64° \ 28'.$

2. Given in the spherical triangle ABC (figs. 4. and 5.)

$$a = 13°, \ c = 9°, \text{ and } B = 176°;$$

to find A and C.

Ans. $A = 2° \ 24',$
$C = 1° \ 40'.$

79. *Theorem. Two spherical triangles have all their sides and angles respectively equal in either of the following cases;* (861)

First. When they have two sides and the included angle respectively equal.

Secondly. When they have one side and the two adjacent angles respectively equal.

Thirdly. When they have their sides respectively equal.

Fourthly. When they have their angles respectively equal.

Demonstration. These propositions are deduced at once from the fact, that the solutions given in articles 36, 38, 46, and 65 led but to one triangle, which (862)
can solve the problem; either, when two sides and the included angle, when a side and the two adjacent angles, when the three sides or the three angles are given.

80. *Theorem. Two spherical right triangles have all their sides and angles equal in the following cases not included in the preceding theorem.* (863)

First. When they have the hypothenuse and one of the angles respectively equal.

Secondly. When they have the hypothenuse and one of the legs respectively equal.

Demonstration. This theorem may be proved in (864)
the same way as the preceding one by a reference to articles 19 and 22.

81. *Scholium.* The case in which the hypothe-
(865) nuse is equal to 90° is, by articles 21 and 24, an exception to the preceding theorem.

CHAPTER IV.

Surfaces of Spherical Triangles.

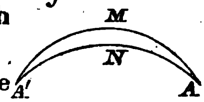
82. All spherical surfaces vary with the radius of
(866) the sphere to which they belong. Consequently some one spherical surface must be assumed as a standard of measure to which they may be referred. We shall assume the surface of the hemisphere as this standard, and shall suppose it to be divided into 360 equal
(867) parts, which we shall call *degrees of surface.* These degrees may be again subdivided into *minutes* and
(868) *seconds.* So that 1° of surface is $\frac{1}{360}$ of the surface of the hemisphere, and 35° of surface are $\frac{35}{360}$ of the surface of the hemisphere.

83. *Definition.* *A lunary surface* is a part of the
(869) surface of a sphere comprehended between two semicircumferences of great circles, which terminate in a common diameter.

(870) 84. *Theorem.* *A lunary surface is measured by double the angle included between its sides.*

Demonstration. Let the lunary surface be $AMNA'$ (fig. 6.), of which the angle is A. (871)

Fig. 6



Then, if the semicircumference AMA' were to depart from coincidence with ANA', and turn round on the diameter AA' till the angle A (872)
had become equal to 180° ; it would have passed over the surface of the hemisphere. And as the sphere is symmetrical, the angle A must increase proportionally with the lunary surface. The lunary surface $AMNA'$ (873)
is therefore the same part of the hemisphere, which the angle A is of 180°, or which $2A$ is of 360°, or

$$\frac{\text{the surface } AMNA'}{\text{surface of hemisphere}} = \frac{2A}{360°};$$ (874)

that is, $2A$ is equal to the number of degrees of surface in $AMNA'$ or is the measure of its surface, as in (875)
(870).

85. *Theorem. Two spherical triangles are equal* (876)
in surface, when their sides and angles are equal.

Demonstration. If these triangles cannot be directly applied to each other, they must be situated as are ABC and DEF in (figs. 7. and 8.); (877)
and it is only in this case that any demonstration is required.

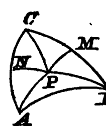
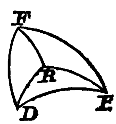


Fig. 7

Draw the arcs MP and NP perpendicular to the middle of the sides BC and AC. From P draw PB, PC, (878)
and PA.



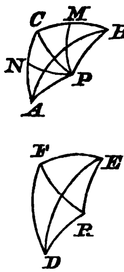


Fig. 8

As the right triangles PMB and PMC have their two legs respec-
(879) tively equal, they must, by (861), give PB equal to PC. and in the same way it may be proved that

(880) $$PA = PC = PB.$$

Again, draw ER and FR, mak-
(881) ing the angles FER and EFR respectively equal to PBC and PCB, and join RD.

The triangles ERF and BCP have the side BC
(882) equal to EF and the two adjacent angles equal, their other sides and angles are therefore, by (861), equal.

The triangles APC and DRF have the sides DF and FR equal to AC and PC, and the included angle

(883) $$ACP = ACB - PCB = DFE - EFR = DFR;$$

their other sides and angles are therefore equal, by (861).

(884) The triangles ABP and DER having their sides equal, must also have their angles equal by (861).

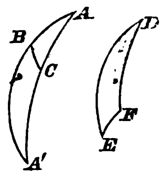
These different triangles can also be respectively
(885) applied to each other, because they are, by (836) and (861), isosceles, and they are therefore equal in surface.

The given triangle ABC being then, in (fig. 7.), the
(886) sum of the three partial triangles ABP, ACP, and BCP, and in (fig. 8.) the difference between ABP and the sum of the other two, must, as in (876), be equal to the triangle DEF, which, in (fig. 7.), is the sum of the three partial triangles DER, DFR, and EFR, and in (fig. 8.) is the difference between DER and the sum of the other two triangles.

86. *Lemma.* If two triangles have an angle of the one equal to an angle of the other; and the sides which include the angle in one triangle are supple- (887)
ments of those which include it in the other triangle; the sum of the surfaces of the two triangles is measured by double the included angle.

Demonstration. Let the triangles be ABC and DEF (fig. 9.), in which A and D are equal; and AB and AC are respectively supplements (888)
of DE and DF.

Fig. 9



Produce AB and AC till they meet in A'. ABA' and ACA' are semicircumferences. In the triangles $A'BC$ and DEF, the angles A' and D are equal, being both equal to A; $A'B$ and DE are equal, being supplements of AB; and $A'C$ and DF are (889)
equal, being supplements of AC. It follows therefore from (861) and (876), that they are equal in surface.

But $A'BC$ and ABC compose the lunary surface $ABCA'$ which is measured by $2\,A$. Therefore the (890)
sum of ABC and DEF is also measured by $2\,A$.

87. *Theorem. The surface of a spherical triangle is measured by the excess of the sum of its three* (891)
angles over two right angles or 180°.

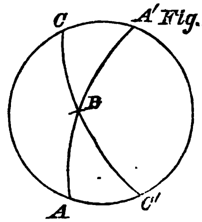


Fig. 10

Demonstration. Let ABC (fig. 10.) be the given triangle. Produce AC to form the circum-
(892) ference $ACA'C'$, also produce AB and BC to form the semicircumferences ABA' and CBC'.

Then, by (870),

(893) the lunary surface $CABC' = 2\ C$,

(894) the lunary surface $ABCA' = 2\ A$;

or

(895) the surface $ABC +$ the surface $ABC' = 2\ C$,

(896) the surface $ABC +$ the surface $A'BC = 2\ A$;

and, by (887),

(897) the surface $ABC +$ the surface $A'BC' = 2\ B$,

for the sides BC and AB are, by (892) supplements
(898) of BC' and $A'B$; and the angle ABC is equal to the angle $A'BC'$.

The sum of (895), (896), and (897), is

(899) $3 \times$ the surface $ABC +$ the surface $A'BC$ $+$ the surface $ABC' +$ the surface $A'BC'$ $= 2\ A + 2\ B + 2\ C$.

But the surface of the hemisphere is, by (867),

(900) the surface $ABC +$ the surface $A'BC$ $+$ the surface $ABC' +$ the surface $A'BC' = 360°$;

which, subtracted from (899), gives

$$2 \times \text{surface } ABC = 2\,A + 2\,B + 2\,C - 360^\circ \qquad (901)$$

or

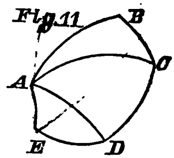
$$\text{surface } ABC = A + B + C - 180^\circ, \qquad (902)$$

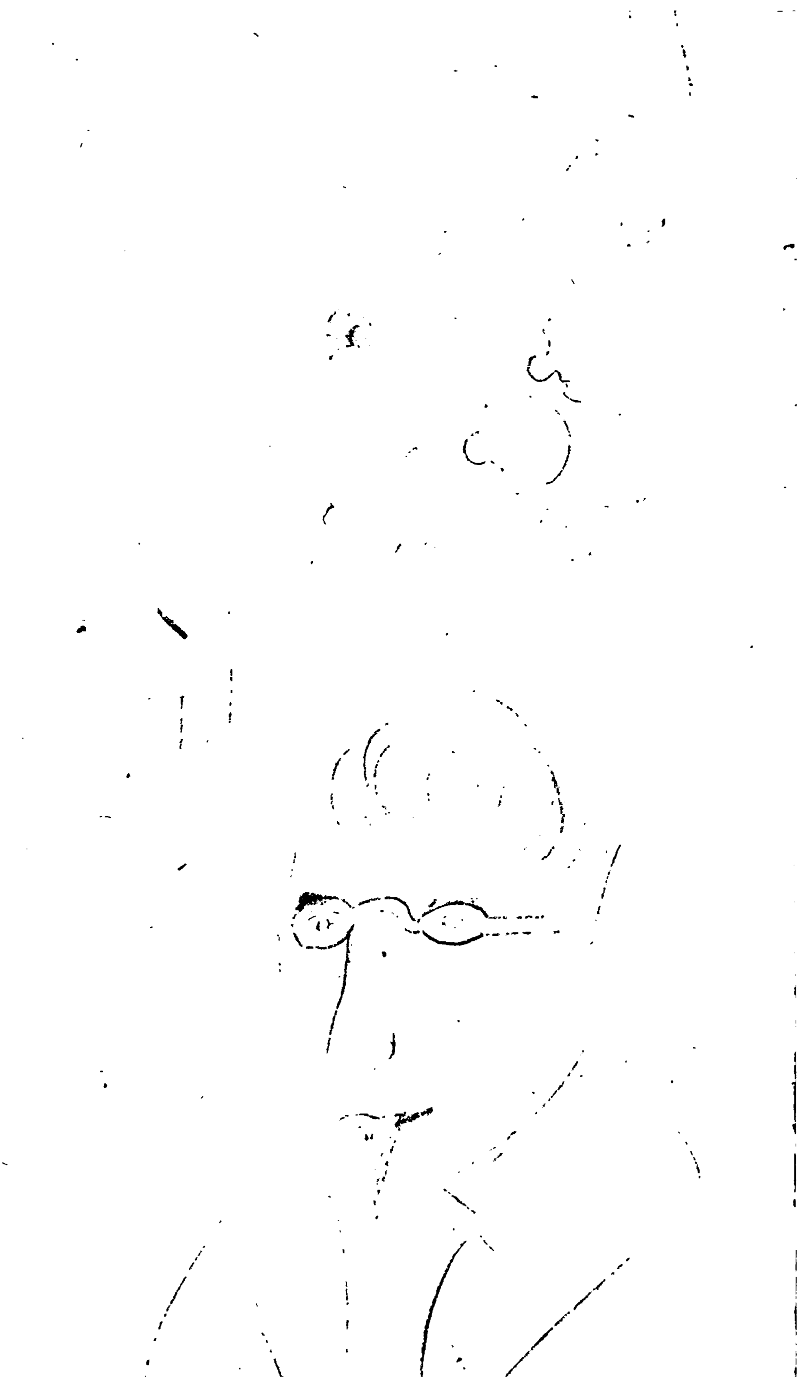
as in (891).

88. *Theorem. The surface of a spherical polygon is equal to the excess of the sum of its angles over as many times two right angles as it has sides minus two.* (903)

Demonstration. Let $ABCDE$ (fig. 11.) be the given polygon. (904) Draw from the vertex A the arcs AC, AD, which divide it into as many triangles as it has sides minus two. (905) By the preceding theorem (891), the sum of the surfaces of all these triangles (906) or the surface of the polygon is equal to the sum of all their angles diminished by as many times two right angles as there are triangles; that is, the surface (907) of the polygon is equal to the sum of all its angles diminished by as many times two right angles, as it has sides minus two, which agrees with (903).

Fig. 11





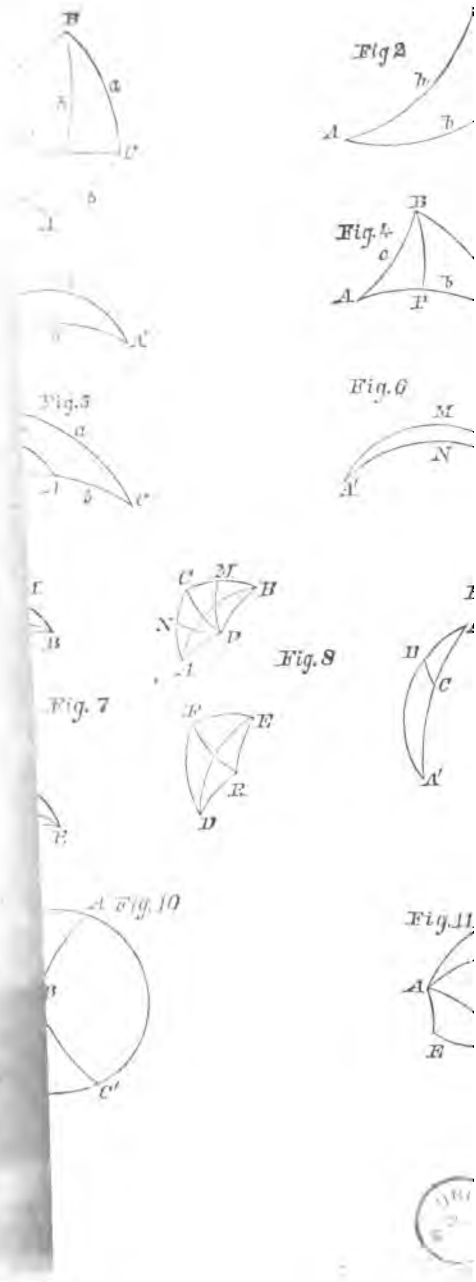


Fig 2
Fig. 4
Fig. 3
Fig. 6
Fig. 8
Fig. 7
Fig. 10
Fig. 11